Matrix Method for Coordinates Transformation

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1. Introduction

Coordinates transformation is a basic part of astronomical calculation and spherical trigonometry has been long used for astronomical calculation in amateur astronomy. Spherical trigonometry equations can be a little bit difficult for amateurs to understand.

In the last two decades, development of personal computers has brought about a change in the way astronomical calculations are carried out. In my opinion, spherical trigonometry is not appropriate to astronomical calculation using personal computers. I recommend the matrix method for coordinates transformation, because of its simplicity and ease of generalization in writing computer programs.

In this monograph, I describe coordinates transformation using the matrix method. I also extend the method to some specific applications, such as polar axis misalignment determination of equatorial mount (Challis' method) and a telescope pointing algorithm.

2. References

[1] Jean Meeus, "Astronomical Formulae for Calculators," 1985, Willmann-Bell, Inc.
[2] Jean Meeus, "Astronomical Algorithms," 1991, Willmann-Bell, Inc.
[3] Ko Nagasawa, "Calculation of Position of Astronomical Objects," 1985, Chijin-Shokan Co., in Japanese
[4] W. R. Vezin, "Polar Axis Alignment of Equatorial Instrument"
[5] Rev. James Challis, "Lectures on Practical Astronomy and Astronomical Instruments," 1879.
[6] Toshimi Taki, "A New Concept in Computer-Aided Telescopes," Sky & Telescope, February 1989, pp.194-196.

3. Notations

3.1 Note

In this monograph, angles are expressed in **radian**, because all computer languages for personal computers use radian for trigonometric functions.

3.2 Symbols

Following symbols are used in this monograph.

X-Y-Z : General rectangular coordinate system

X_e-Y_e-Z_e : Rectangular equatorial coordinate system

X_h-Y_h-Z_h : Rectangular altazimuth coordinate system

X_t-Y_t-Z_t : Rectangular telescope coordinate system

a : Right Ascension (in radian)

d : Declination (in radian)

A : Azimuth, measured westward from the South. (in radian)

h : Altitude (in radian)

x : General polar coordinate measured counterclockwise from X-axis in XY-plane (in radian)

z : General polar coordinate, measured upward from XY-plane. (in radian)

j , q : Telescope polar coordinates (in radian)

D, D', D'' : Telescope mount fabrication errors (in radian)

L : X-component of direction cosine of celestial object in X-Y-Z coordinates

M : Y-component of direction cosine of celestial object in X-Y-Z coordinates

N : Z-component of direction cosine of celestial object in X-Y-Z coordinates

H : Hour angle

f : Observer's latitude

q_0 : Sidereal time at Greenwich

t : Time

u, v : Telescope polar axis misalignment (in radian)

JD : Julian day number

d : Angular distance between two objects

$[T]$: Transformation matrix between coordinate systems

R : Atmospheric refraction (in radian)

4. Basic Equations of Coordinates Transformation in Matrix Method

4.1 Polar Coordinates and Rectangular Coordinates

In astronomical calculations, polar coordinate systems are usually used. See figure 4-1. Point O is the observation point. Vector OR shows unit vector directing to a celestial object. The position of the celestial object is express in polar coordinates (x, z). Normally, angle x is measured counterclockwise from X-axis (viewing from positive Z) and angle z is measured upward (toward Z-axis) from XY-plane.

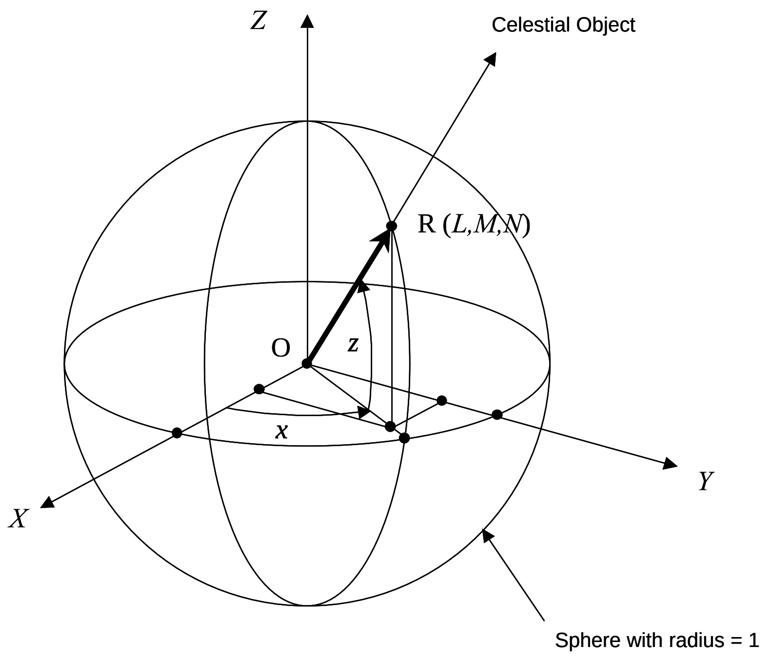


Figure 4-1 Polar Coordinates and Rectangular Coordinates

An example of polar coordinates is right ascension and declination, (a, d). See figure 4-2.

The other example is azimuth and altitude, (A, h). But azimuth is measured westward (clockwise) from the South which is the opposite direction to the normal polar coordinate system. See figure 4-3.

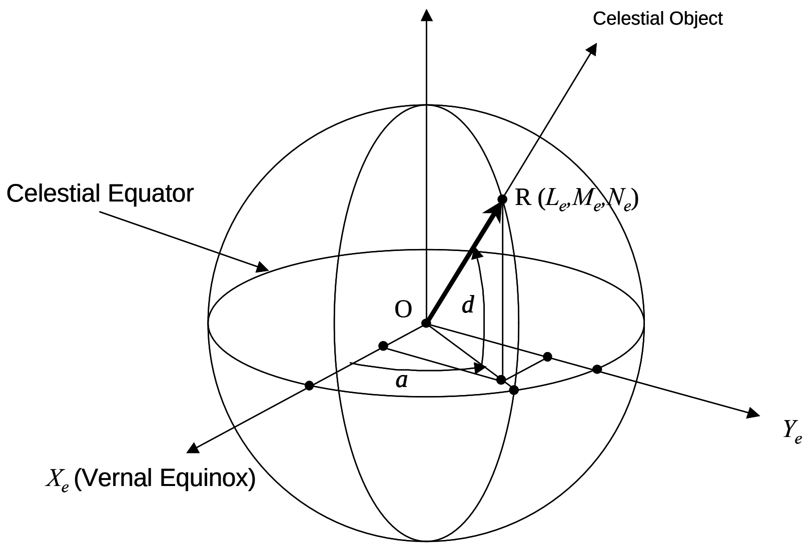


Figure 4-2 Equatorial Coordinates

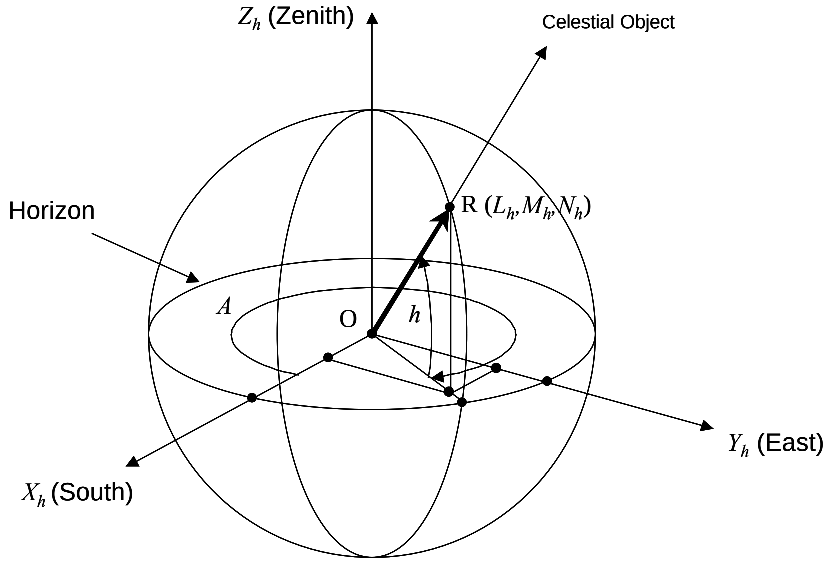


Figure 4-3 Altazimuth Coordinates

The vector OR is also expressed in rectangular coordinates, (L, M, N). (L, M, N) is called direction cosine. In the matrix method, direction cosines are used to express coordinate transformation.

Relationship between rectangular coordinates and polar coordinates can be expressed in matrix form as follows.

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \cos z \cos x \\ \cos z \sin x \\ \sin z \end{pmatrix}$$ …. Equation (4-1)

For equatorial coordinates,

$$\begin{pmatrix} L_e \\ M_e \\ N_e \end{pmatrix} = \begin{pmatrix} \cos d \cos a \\ \cos d \sin a \\ \sin d \end{pmatrix}$$ …. Equation (4-2)

For horizontal coordinates,

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{pmatrix} \cos h \cos(-A) \\ \cos h \sin(-A) \\ \sin h \end{pmatrix}$$ …. Equation (4-3)

Note that ($-A$) is used in the equation (4-3) instead of A, because azimuth A is measured clockwise.

4.2 Coordinate Transformation

4.2.1 New Coordinate System Rotated around Z-axis

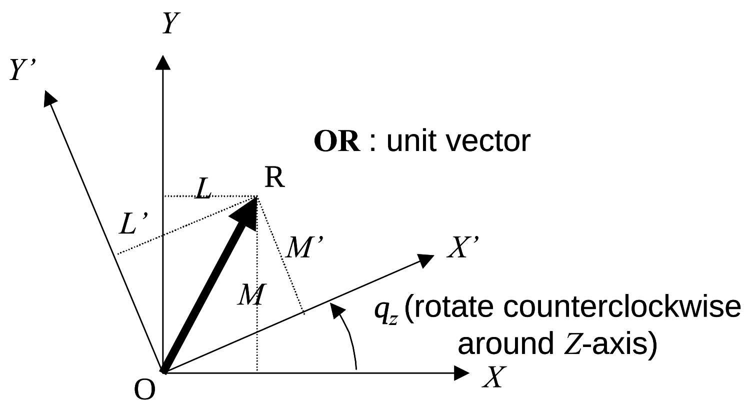
New coordinate system, *X'-Y'-Z'* is generated rotating *X-Y-Z* coordinates around *Z*-axis as shown in figure 4-4.

The polar coordinates in *X'-Y'-Z'* coordinate system is *(x', z')* and the direction cosine in *X'-Y'-Z'* coordinate system is (*L', M', N'*). The relationship between the direction cosines in both coordinate systems is expressed as follows.

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = \begin{pmatrix} \cos z' \cos x' \\ \cos z' \sin x' \\ \sin z' \end{pmatrix} \quad \text{.... Equation (4-4)}$$

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = \begin{bmatrix} \cos q_z & \sin q_z & 0 \\ -\sin q_z & \cos q_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix} \text{.... Equation (4-5)}$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{bmatrix} \cos q_z & -\sin q_z & 0 \\ \sin q_z & \cos q_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} \text{.... Equation (4-6)}$$



Looking Normal to *XY*-plane

Figure 4-4 Coordinates Rotation around Z-axis

4.2.2 New Coordinate System Rotated around X-axis

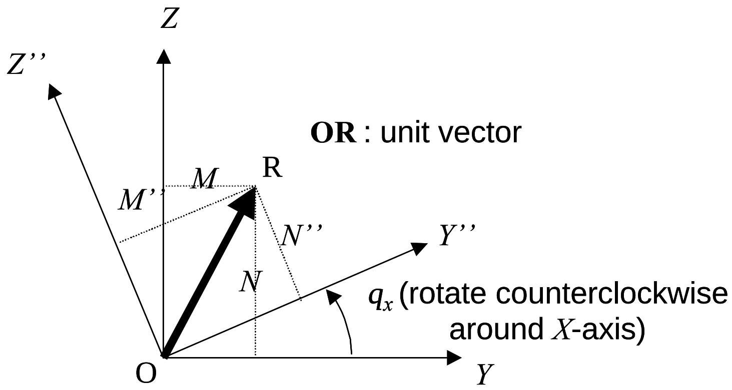
New coordinate system, *X”-Y”-Z”* is generated rotating *X-Y-Z* coordinates around *X*-axis as shown in figure 4-5.

The polar coordinates in *X”-Y”-Z”* coordinate system is *(x”, z”)* and the direction cosine in *X”-Y”-Z”* coordinate system is (*L”, M”, N”*). Then the relationship between the direction cosines in both coordinate systems is expressed as follows.

$$\begin{pmatrix} L'' \\ M'' \\ N'' \end{pmatrix} = \begin{pmatrix} \cos z''\cos x'' \\ \cos z''\sin x'' \\ \sin z'' \end{pmatrix} \quad \text{.... Equation (4-7)}$$

$$\begin{pmatrix} L'' \\ M'' \\ N'' \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_x & \sin q_x \\ 0 & -\sin q_x & \cos q_x \end{bmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix} \quad \text{.... Equation (4-8)}$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_x & -\sin q_x \\ 0 & \sin q_x & \cos q_x \end{bmatrix} \begin{pmatrix} L'' \\ M'' \\ N'' \end{pmatrix} \quad \text{.... Equation (4-9)}$$



Looking Normal to *YZ*-plane

Figure 4-5 Coordinates Rotation around X-axis

4.2.3 New Coordinate System Rotated around Y-axis

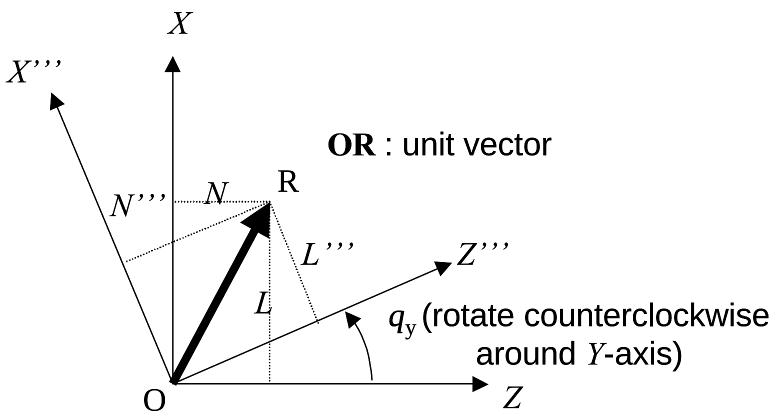
New coordinate system, *X'''-Y'''-Z'''* is generated rotating *X-Y-Z* coordinates around *Y*-axis as shown in figure 4-6.

The polar coordinates in *X'''-Y'''-Z'''* coordinate system is *(x''', z''')* and the direction cosine in *X'''-Y'''-Z'''* coordinate system is (*L''', M''', N'''*). Then the relationship between the direction cosines in both coordinate systems is expressed as follows.

$$\begin{pmatrix} L''' \\ M''' \\ N''' \end{pmatrix} = \begin{pmatrix} \cos z'''\cos x'''' \\ \cos z''''\sin x'''' \\ \sin z'''' \end{pmatrix}$$ …. Equation (4-10)

$$\begin{pmatrix} L''' \\ M''' \\ N''' \end{pmatrix} = \begin{bmatrix} \cos q_y & 0 & -\sin q_y \\ 0 & 1 & 0 \\ \sin q_y & 0 & \cos q_y \end{bmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$ …. Equation (4-11)

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{bmatrix} \cos q_y & 0 & \sin q_y \\ 0 & 1 & 0 \\ -\sin q_y & 0 & \cos q_x \end{bmatrix} \begin{pmatrix} L''' \\ M''' \\ N''' \end{pmatrix}$$ …. Equation (4-12)



Looking Normal to *ZX*-plane

Figure 4-6 Coordinates Rotation around Y-axis

4.3 Obtaining Polar Coordinates from Direction Cosine

After coordinate transformation using the matrix method it is necessary to obtain the polar coordinates (x', z') from the direction cosines.

Using equation (4-4), x' and z' are obtained from direction cosines as shown below.

$$\tan x' = \frac{M'}{L'} \quad \text{.... Equation (4-13)}$$

When $L' >= 0$, x' is in the 1st quadrant or the 4th quadrant.

When $L' < 0$, x' is in the 2nd quadrant or the 3rd quadrant.

$$\sin z' = N' \quad \text{.... Equation (4-14)}$$

$-p/2\ (-90^{o}) <= z' <= +p/2\ (+90^{o})$

4.4 Notes on Approximation

4.4.1 Approximation of Trigonometric Functions

When we process small angles in trigonometry, approximation of trigonometric functions is often used.

In the following approximations, q is very small angle and expressed in radian.

$\sin q \cong q$ …. Equation (4-15)

$\cos q \cong 1$ …. Equation (4-16)

For higher order approximation,

$\cos q \cong 1 - \frac{q^2}{2}$ …. Equation (4-17)

4.4.2 Approximation of Other Functions

For other functions, following approximation can be used when x is very small compared to 1.

$\sqrt{1+x} \cong 1 + \frac{x}{2}$ …. Equation (4-18)

$(1+x)^2 \cong 1 + 2x$ …. Equation (4-19)

5. Applications

5.1 Transformation from Equatorial Coordinates to Altazimuth Coordinates

5.1.1 Transformation Equations

Altazimuth coordinate system, X_h-Y_h-Z_h is rotated $-(p/2 - f)$ around Y_h-axis to equatorial coordinate system, X_e'-Y_e'-Z_e'. f is observer's latitude. See figure 5.1-1.

The direction cosines are expressed in angles as follows.

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{pmatrix} \cos h \cos(-A) \\ \cos h \sin(-A) \\ \sin h \end{pmatrix}$$ …. Equation (5.1-1)

Where A is azimuth measured westward from the South and h is altitude.

$$\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix}$$ …. Equation (5.1-2)

Where H is local hour angle measure westward from the South and d is declination.

Relationship between the coordinates is expressed in matrix form as shown below.

$$\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & -\sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix}$$ …. Equation (5.1-3)

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & \sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix}$$ …. Equation (5.1-4)

$$\tan(-A) = \frac{M_h}{L_h}$$ …. Equation (5.1-5)

When $L_h >= 0$, $(-A)$ is in the 1st quadrant or the 4th quadrant.

When $L_h < 0$, $(-A)$ is in the 2nd quadrant or the 3rd quadrant.

$$\sin h = N_h$$ …. Equation (5.1-6)

$-p/2\ (-90^o) <= h <= +p/2\ (+90^o)$

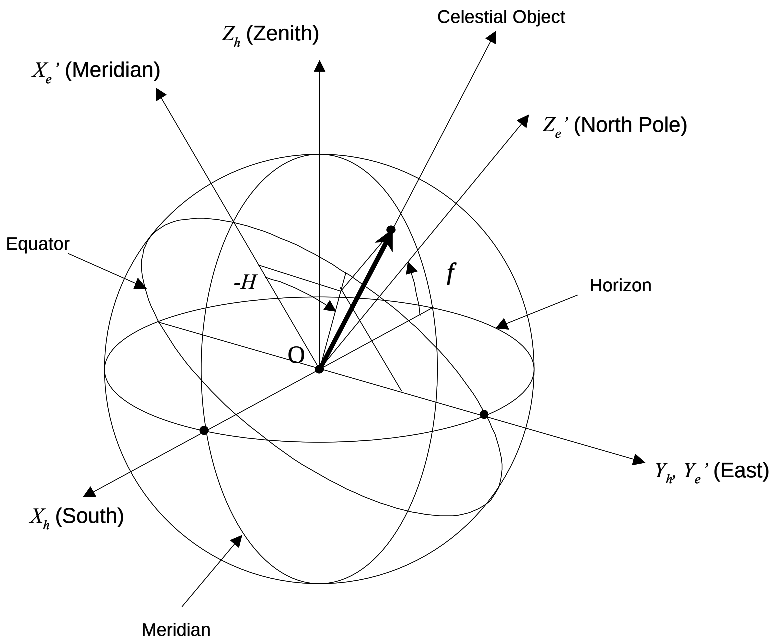


Figure 5.1-1 Altazimuth Coordinates and Equatorial Coordinates

Comparison with spherical trigonometric equations (ref. [1]) is performed below.
From equations (5.1-2), (5.1-4) and (5.1-5), we obtain the following equations.

$$\begin{aligned}
\tan(-A) &= \frac{M_e'}{\cos\left(f-\frac{p}{2}\right)L_e' + \sin\left(f-\frac{p}{2}\right)N_e'} \\
&= \frac{\sin(-H)\cos d}{\cos\left(f-\frac{p}{2}\right)\cos(-H)\cos d + \sin\left(f-\frac{p}{2}\right)\sin d} \\
&= \frac{\sin(-H)}{\cos\left(f-\frac{p}{2}\right)\cos(-H) + \sin\left(f-\frac{p}{2}\right)\tan d} \\
&= \frac{\sin(-H)}{\sin f\cos H - \cos f\tan d}
\end{aligned}$$

…. Equation (5.1-7)

$$\sin h = -\sin\left(f - \frac{p}{2}\right)\cos(-H)\cos d + \cos\left(f - \frac{p}{2}\right)\sin d$$
$$= \cos f \cos H \cos d + \sin f \sin d$$

…. Equation (5.1-8)

These equations are the same as equations (8.5) and (8.6) in ref. [1].

5.1.2 Example Calculation

Example 8.b in ref. [1]:

Find the azimuth and the altitude of Saturn on 1978 November 13 at 4h34m00s UT at the Uccle Observatory (longitude –0h17m25.94s, latitude +50°47’55.0” = 0.88660302 (radian)); the planet’s apparent equatorial coordinates, interpolated from the A.E., being

a = 10h57m35.681s = 10.9599114h = 2.86929809 (radian)

d = +8°25’58.10” = 8.432806° = 0.14718022 (radian)

The apparent sidereal time at Greenwich, q_0 = 8h01m46.135s.

Local hour angle, H is,

$H = q_0 - L - a$

= 8h01m46.135s + 0h17m25.94s – 10h57m35.681s

= -2h38m23.606s

= -2.6398906h

= -2.6398906 x 15 / (180/π) … (radian)

= -0.69112174 (radian)

From equation (5.1-2),

$$\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos(-H)\cos d \\ \sin(-H)\cos d \\ \sin d \end{pmatrix} = \begin{pmatrix} 0.76220092 \\ 0.63051067 \\ 0.14664943 \end{pmatrix}$$

From equation (5.1-4),

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & \sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix}$$

$$= \begin{bmatrix} \cos(-0.68419331) & 0 & \sin(-0.68419331) \\ 0 & 1 & 0 \\ -\sin(-0.68419331) & 0 & \cos(-0.68419331) \end{bmatrix} \begin{pmatrix} 0.76220092 \\ 0.63051067 \\ 0.14664943 \end{pmatrix}$$

$$= \begin{bmatrix} 0.77492917 & 0 & -0.63204809 \\ 0 & 1 & 0 \\ 0.63204809 & 0 & 0.77492917 \end{bmatrix} \begin{pmatrix} 0.76220092 \\ 0.63051067 \\ 0.14664943 \end{pmatrix}$$

$$= \begin{pmatrix} 0.49796223 \\ 0.63051067 \\ 0.59539056 \end{pmatrix}$$

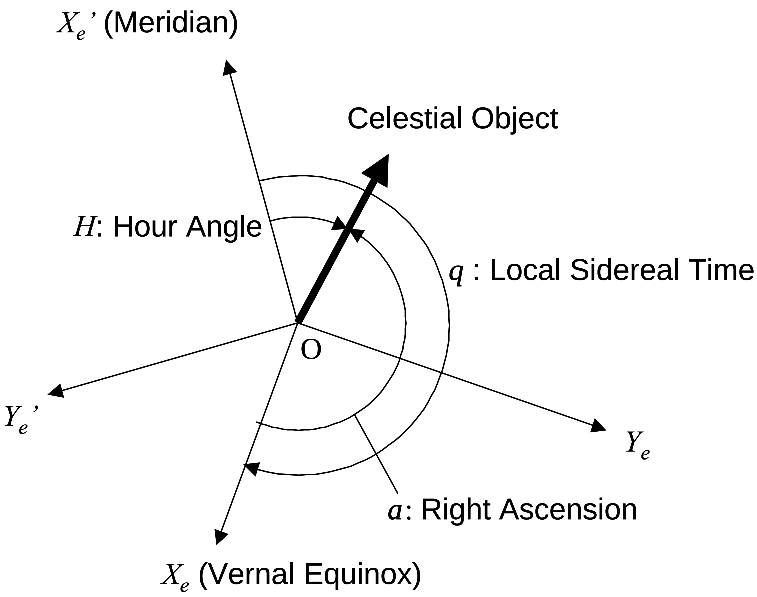
From equations (5.1-5) and (5.1-6),

$$\tan(-A) = \frac{M_h}{L_h} = \frac{0.63051067}{0.49796223} = 1.2661817$$

$-A$ = 0.90232066 (radian) → $\boldsymbol{A}$ = -0.9032066 (radian) = **-51.6992^{o}**

$\sin h$ = 0.59539056

$\boldsymbol{h}$ = 0.63775167 (radian) = **36.5405^{o}**



Looking from North Pole Normal to Equatorial Plane

Figure 5.1-2 Hour Angle and Sidereal Time

5.2 Angular Separation

5.2.1 Equations

The angular distance d between two celestial objects, P1 and P2 is derived using the matrix method.

Position of object 1, P_1: (x_1, z_1)

Position of object 2, P_2: (x_2, z_2)

Direction cosines of the two objects are,

$$\begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix} = \begin{pmatrix} \cos z_1 \cos x_1 \\ \cos z_1 \sin x_1 \\ \sin z_1 \end{pmatrix} \quad \text{…. Equation (5.2-1)}$$

$$\begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix} = \begin{pmatrix} \cos z_2 \cos x_2 \\ \cos z_2 \sin x_2 \\ \sin z_2 \end{pmatrix} \quad \text{…. Equation (5.2-2)}$$

Using scalar product of the two unit vectors, $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, angular separation d is obtained as follows. See figure 5.2-1.

$$\begin{aligned} \cos d &= L_1 L_2 + M_1 M_2 + N_1 N_2 \\ &= \cos x_1 \cos x_2 \cos V_1 \cos V_2 + \sin x_1 \sin x_2 \cos V_1 \cos V_2 + \sin V_1 \sin V_2 \\ &= \sin V_1 \sin V_2 + \cos V_1 \cos V_2 \cos(x_1 - x_2) \end{aligned}$$

…. Equation (5.2-3)

This equation is identical to equation (9.1) in ref. [1].

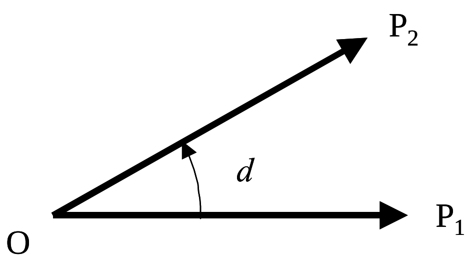


Figure 5.2-1 Angular Separation

When angular separation is very small, (ξ_1-ξ_2) and (ζ_1-ζ_2) are nearly zero and equation (5.2-2) can not be used. Equation (5.2-2) is transformed to a new equation as follows.

$$\cos d = \sin V_1 \sin V_2 + \cos V_1 \cos V_2 \cos \Delta x$$
$$\cong \sin V_1 \sin V_2 + \cos V_1 \cos V_2 \left(1 - \frac{(\Delta x)^2}{2}\right)$$
$$= \sin V_1 \sin V_2 + \cos V_1 \cos V_2 - \frac{(\Delta x)^2}{2} \cos V_1 \cos V_2$$
$$= \cos \Delta V - \frac{(\Delta x)^2}{2} \cos^2 V$$
$$\cong 1 - \frac{(\Delta V)^2}{2} - \frac{(\Delta x)^2}{2} \cos^2 V$$

Where $V = \frac{V_1 - V_2}{2}$

$$\cos d$$
$$\cong 1 - \frac{(\Delta d)^2}{2} \cong 1 - \frac{(\Delta V)^2}{2} - \frac{(\Delta x)^2}{2} \cos^2 V$$

$$(\Delta d)^2 \cong (\Delta V)^2 - (\Delta x)^2 \cos^2 V$$

$$d = \sqrt{(\Delta x \cdot \cos V)^2 + (\Delta V)^2}$$ …. Equation (5.2-4)

Note: When θ is very small, the following approximation can be used.

$$\cos q \cong 1 - \frac{q^2}{2}$$ (in radian)

5.2.2 Example Calculation

Example 9.a in ref. [1]:

Calculate the angular distance, *d* between Arcturus and Spica.

The 1950 coordinates of these stars are,

Arcturus :	a_1 = 14h13m22.8s = 213.3450° = 3.72357269	d_1 = +19°26’31” = 0.33932594
Spica :	a_2 = 13h22m33.3s = 200.6388° = 3.50180767	d_2 = -10°54’03” = -0.19025543

$$\begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix} = \begin{pmatrix} \cos\boldsymbol{d}_1 \cos\boldsymbol{a}_1 \\ \cos\boldsymbol{d}_1 \sin\boldsymbol{a}_1 \\ \sin\boldsymbol{d}_1 \end{pmatrix} = \begin{pmatrix} -.78774214 \\ -0.51833597 \\ 0.33285154 \end{pmatrix}$$

$$\begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix} = \begin{pmatrix} \cos\boldsymbol{d}_2 \cos\boldsymbol{a}_2 \\ \cos\boldsymbol{d}_2 \sin\boldsymbol{a}_2 \\ \sin\boldsymbol{d}_2 \end{pmatrix} = \begin{pmatrix} -0.91893507 \\ -0.34611538 \\ -0.18910972 \end{pmatrix}$$

$$\begin{aligned} \cos d &= L_1 L_2 + M_1 M_2 + N_1 N_2 \\ &= -0.78774214 \times (-0.91893507) + (-0.51833597) \times (-0.34611538) \\ &\qquad + 0.33285154 \times (-0.18910972) \\ &= 0.84034247 \end{aligned}$$

$\boldsymbol{d}$ = 0.57288162 (radian) = $\mathbf{32.8237^o}$

5.3 Compensation of Mounting Fabrication Errors

5.3.1 Telescope Coordinates

A telescope has telescope coordinate system as shown in figure 5.3-1. True telescope polar coordinates is (j, q). “True” means that we consider hypothetical perfect telescope mount without fabrication error.

If the X_t-axis points to the South and Z_t-axis points to zenith, this mount is an alt-azimuth mount.

$$j = -A$$
$$q = h$$

If the Z_t-axis points to celestial north pole and X_t-axis points to meridian, this mount is an equatorial mount.

$$j = -H$$
$$q = d$$

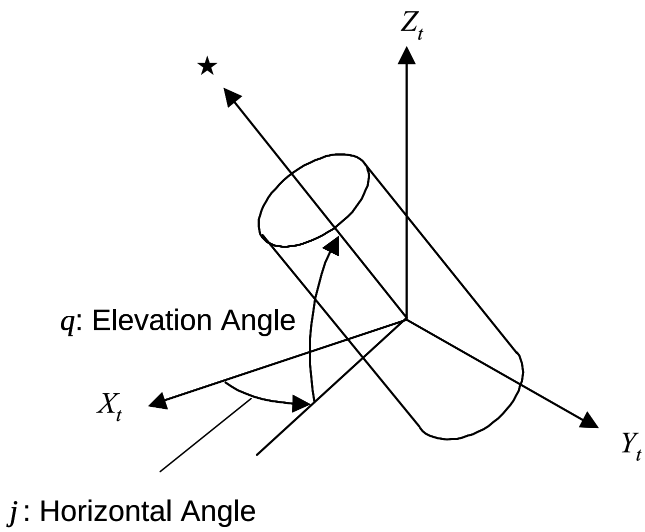


Figure 5.3-1 Telescope Coordinates

5.3.2 Fabrication Errors of Mount

In the real world all mountings have fabrication errors. There are three different fabrication errors to be considered as shown in figure 5.3-2.

(1) D: Error in perpendicularity between horizontal axis and vertical axis, or polar axis

and declination axis

(2) D': Collimation error between vertical or polar axis and telescope optical axis

(3) D'': Shift of zero point in apparent elevation angle or declination angle

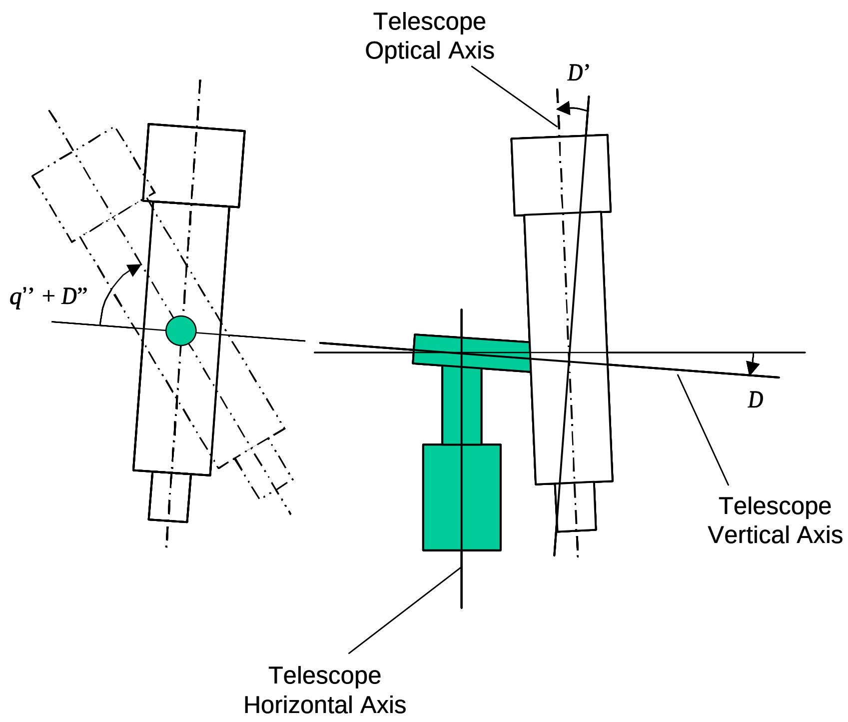


Figure 5.3-2 Telescope Mount Fabrication Error

5.3.3 Derivation of Equations

The apparent telescope coordinates (j', q') is the coordinates measured with setting circles of the telescope mount. Relationship between the true telescope coordinates and the apparent telescope coordinates are derived as follows. See figures 5.2-3 to 5.2-5.

(1) Telescope optical axis, R'''-axis points to a celestial object of true telescope coordinates (j, q). $R'''S'''$-plane is the plane defined by telescope optical axis and telescope vertical axis. This means that direction cosines of the celestial object in R'''-S'''-T''' coordinates is $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

(2) The coordinate system *R'''-S'''-T'''* is rotated *–D'* counterclockwise around T'''-axis and becomes a new coordinate system *R''-S''-T''*.

(3) The coordinate system R''-S''-T'' is rotated (*q'+D''*) counterclockwise around *S''*-axis and becomes *R'-S'-T'* coordinate system.

(4) The coordinate system *R'-S'-T'* is rotated *-D* counterclockwise around R'-axis and becomes *R-S-T* coordinate system.

(5) Finally, the coordinate system *R-S-T* is rotated *–j'* counterclockwise around T-axis and becomes *X-Y-Z* coordinate system which is the true telescope coordinates.

$$
\begin{pmatrix} \cos q \cos j \\ \cos q \sin j \\ \sin q \end{pmatrix} = \begin{bmatrix} \cos j' & -\sin j' & 0 \\ \sin j' & \cos j' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Delta & -\sin\Delta \\ 0 & \sin\Delta & \cos\Delta \end{bmatrix}
$$

$$
\times \begin{bmatrix} \cos(q'+\Delta'') & 0 & -\sin(q'+\Delta'') \\ 0 & 1 & 0 \\ \sin(q'+\Delta'') & 0 & \cos(q'+\Delta'') \end{bmatrix} \begin{bmatrix} \cos\Delta' & -\sin\Delta' & 0 \\ \sin\Delta' & \cos\Delta' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos(q'+\Delta'')\cos j'\cos\Delta' - \sin j'\cos\Delta\sin\Delta' + \sin(q'+\Delta'')\sin j'\sin\Delta\cos\Delta' \\ \cos(q'+\Delta'')\sin j'\cos\Delta' + \cos j'\cos\Delta\sin\Delta' - \sin(q'+\Delta'')\cos j'\sin\Delta\cos\Delta' \\ \sin(q'+\Delta'')\cos\Delta\cos\Delta' + \sin\Delta\sin\Delta' \end{pmatrix}
$$

…. Equation (5.3-1)

Equation (5.3-1) is an exact solution to obtain true telescope coordinate from apparent telescope coordinate.

Using the following approximation,

$$
\sin j'\cos\Delta\sin\Delta' \approx \sin j \cos\Delta\sin\Delta', \quad \sin(q'+\Delta'')\sin j'\sin\Delta\cos\Delta' \approx \sin q \sin j \sin\Delta\cos\Delta'
$$

$$
\cos j'\cos\Delta\sin\Delta' \approx \cos j \cos\Delta\sin\Delta', \quad \sin(q'+\Delta'')\cos j'\sin\Delta\cos\Delta' \approx \sin q \cos j \sin\Delta\cos\Delta'
$$

Equation (5.3-2) is derived from equation (5.3-1).

$$
\begin{pmatrix} \cos(q'+\Delta'')\cos j' \\ \cos(q'+\Delta'')\sin j' \\ \sin(q'+\Delta'') \end{pmatrix} = \begin{pmatrix} (\cos q \cos j + \sin j \cos\Delta\sin\Delta' - \sin q \sin j \sin\Delta\cos\Delta') / \cos\Delta' \\ (\cos q \sin j - \cos j \cos\Delta\sin\Delta' + \sin q \cos j \sin\Delta\cos\Delta') / \cos\Delta' \\ (\sin q - \sin\Delta\sin\Delta') / \cos\Delta\cos\Delta' \end{pmatrix}
$$

…. Equation (5.3-2)

Using equation (5.3-2), an exact solution of q' and an approximate solution of j' are obtained.

Further approximation can be made as follows.
Assuming the errors are very small, equations (5.3-1) and (5.3-2) are simplified as follows.

$$\begin{pmatrix} \cos q \cos j \\ \cos q \sin j \\ \sin q \end{pmatrix} = \begin{pmatrix} \cos(q'+\Delta'')\cos j' - \Delta'\sin j' + \Delta\sin(q'+\Delta'')\sin j' \\ \cos(q'+\Delta'')\sin j' + \Delta'\cos j' - \Delta\sin(q'+\Delta'')\cos j' \\ \sin(q'+\Delta'') \end{pmatrix}$$

…. Equation (5.3-3)

$$\begin{pmatrix} \cos(q'+\Delta'')\cos j' \\ \cos(q'+\Delta'')\sin j' \\ \sin(q'+\Delta'') \end{pmatrix} = \begin{pmatrix} \cos q \cos j + \Delta'\sin j - \Delta\sin q \sin j \\ \cos q \sin j - \Delta'\cos j + \Delta\sin q \cos j \\ \sin q \end{pmatrix}$$

…. Equation (5.3-4)

5.3.4 Apparent Telescope Coordinate without Approximation

In order to obtain an exact solution of apparent telescope coordinate j', iteration will be made as follows.

Rewriting equation (5.3-1), we get the following equation.

$$\begin{pmatrix} \cos(q'+\Delta'')\cos j' \\ \cos(q'+\Delta'')\sin j' \\ \sin(q'+\Delta'') \end{pmatrix} = \begin{pmatrix} (\cos q \cos j + \sin j'\cos\Delta\sin\Delta' - \sin(q'+\Delta'')\sin j'\sin\Delta\cos\Delta')/\cos\Delta' \\ (\cos q \sin j - \cos j'\cos\Delta\sin\Delta' + \sin(q'+\Delta'')\cos j'\sin\Delta\cos\Delta')/\cos\Delta' \\ (\sin q - \sin\Delta\sin\Delta')/\cos\Delta\cos\Delta' \end{pmatrix}$$

$$= \begin{pmatrix} (\cos q \cos j + \sin j'\cos\Delta\sin\Delta' - (\sin q - \sin\Delta\sin\Delta')\sin j'\sin\Delta/\cos\Delta)/\cos\Delta' \\ (\cos q \sin j - \cos j'\cos\Delta\sin\Delta' + (\sin q - \sin\Delta\sin\Delta')\cos j'\sin\Delta/\cos\Delta)/\cos\Delta' \\ (\sin q - \sin\Delta\sin\Delta')/\cos\Delta\cos\Delta' \end{pmatrix}$$

…. Equation (5.3-5)

Using equation (5.3-2), the first approximate solution (j'_1, q') is obtained. The first approximate solution is input into equation (5.3-5),

$$
\begin{pmatrix} \cos(q'+\Delta'')\cos j'_2 \\ \cos(q'+\Delta'')\sin j'_2 \\ \sin(q'+\Delta'') \end{pmatrix}
= \begin{pmatrix} (\cos q \cos j + \sin j'_1 \cos\Delta \sin\Delta' - (\sin q - \sin\Delta\sin\Delta')\sin j'_1 \sin\Delta/\cos\Delta)/\cos\Delta' \\ (\cos q \sin j - \cos j'_1 \cos\Delta \sin\Delta' + (\sin q - \sin\Delta\sin\Delta')\cos j'_1 \sin\Delta/\cos\Delta)/\cos\Delta' \\ (\sin q - \sin\Delta\sin\Delta')/\cos\Delta\cos\Delta' \end{pmatrix}
$$

…. Equation (5.3-6)

Solving this equation, the second approximate solution (j'_2, q') is obtained.

This iteration will be performed until the solution converges. If the mount fabrication errors are about 1 degree, two iterations are enough.

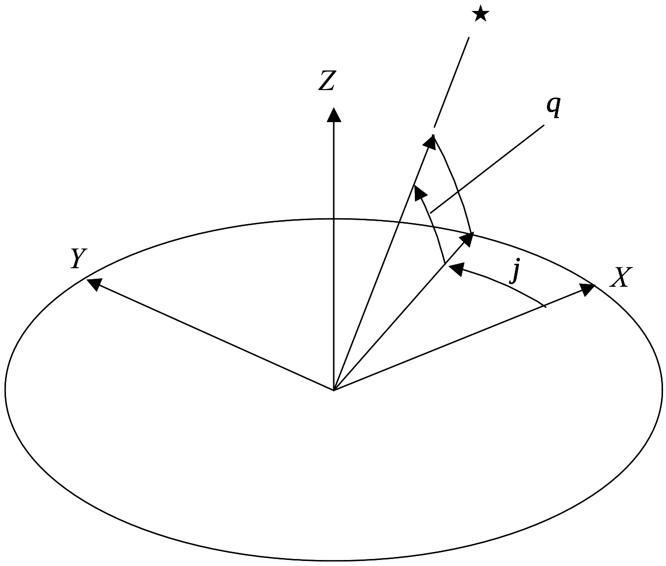


Figure 5.3-3　True Telescope Coordinates

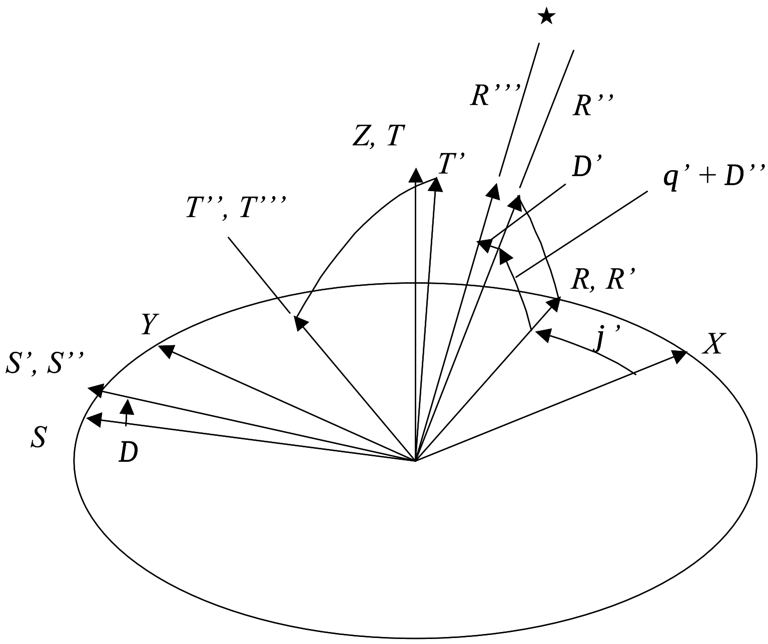


Figure 5.3-4　Apparent Telescope Coordinate

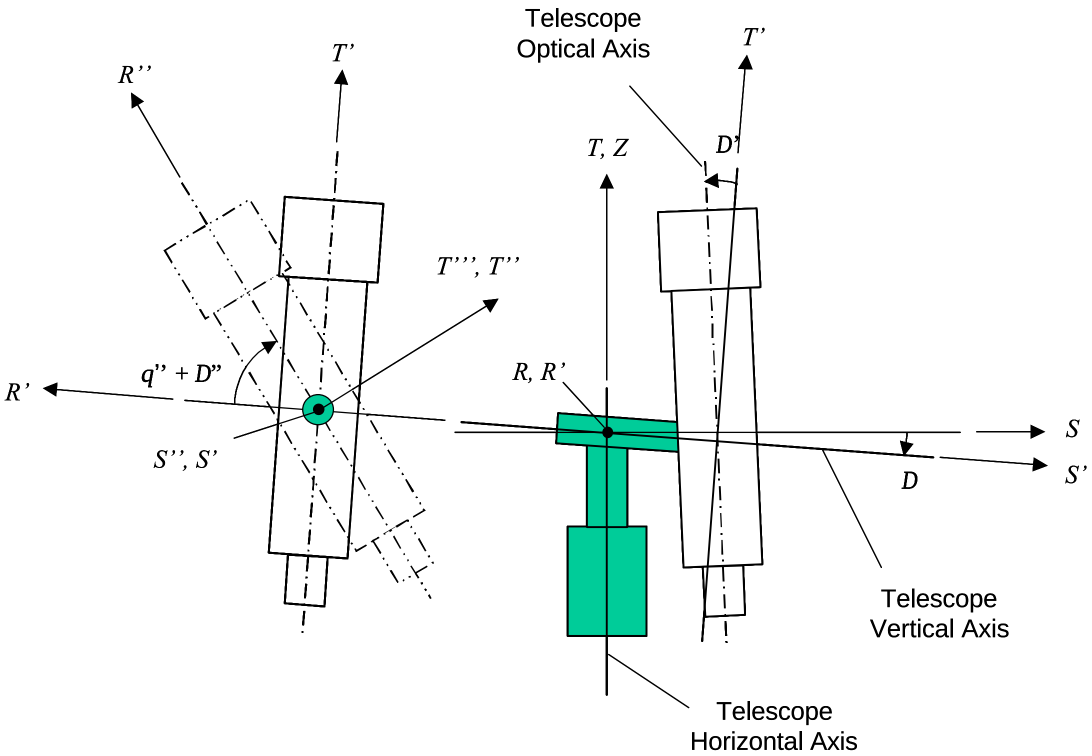


Figure 5.3-5 Apparent Telescope Coordinates with Mount Error

5.3.5 Example Calculations

5.3.5.1 Apparent Telescope Coordinates → True Telescope Coordinates

Find true telescope coordinates from apparent telescope coordinates.

(1) Data

Mount errors are given as shown below.

$D = 0.15^{o} = 0.15 / 180 \times p = 0.0026179939$ (radian)

$D' = -0.08^{o} = -0.08 / 180 \times p = -0.0013962634$ (radian)

$D'' = 0.2^{o} = 0.2 / 180 \times p = 0.0034906585$ (radian)

Measured position (apparent telescope coordinates) of a celestial object, (q', j') is,

$q' = 62.3000^{o} = 62.3 / 180 \times p = 1.08734012$ (radian)

$j' = 53.5000^{o} = 53.5 / 180 \times p = 0.93375115$ (radian)

(2) Calculation

$q' + D'' = 1.08734012 + 0.0034906585 = 1.09083078$ (radian)

Exact solution is obtained from equation (5.3-1),

$$\begin{pmatrix} \cos q \cos j \\ \cos q \sin j \\ \sin q \end{pmatrix}$$

$$= \begin{pmatrix} \cos 1.09083078 \cos 0.93375115 \cos(-0.0013962634) \\ \quad - \sin(-0.0013962634) \cos 0.0026179939 \sin 0.93375115 \\ \quad\quad + \sin 1.09083078 \sin 0.93375115 \sin 0.0026179939 \cos(-0.0013962634) \\ \\ \cos 1.09083078 \sin 0.93375115 \cos(-0.0013962634) \\ \quad + \sin(-0.0013962634) \cos 0.0026179939 \cos 0.93375115 \\ \quad\quad - \sin 1.09083078 \cos 0.93375115 \sin 0.0026179939 \cos(-0.0013962634) \\ \\ \sin 1.09083078 \cos 0.0026179939 \cos(-0.0013962634) \\ \quad\quad + \sin 0.0026197739 \sin(-0.0013962634) \end{pmatrix}$$

$$= \begin{pmatrix} 0.27764743 \\ 0.36896762 \\ 0.88700327 \end{pmatrix}$$

From equations (4-13) and (4-14),

$$\tan j = \frac{M}{L} = \frac{0.36896762}{0.27764743} = 1.32890702$$

$\boldsymbol{j}$ = 0.92569835 (radian) = **53.0386°**

$\sin q$ = 0.88700327

$\boldsymbol{q}$ **=** 1.09081440 (radian) = **62.4991°**

Approximate solution is obtained from equation (5.3-4) as follows.

$$\begin{pmatrix} \cos q \cos j \\ \cos q \sin j \\ \sin q \end{pmatrix}$$

$$= \begin{pmatrix} \cos 1.09083078 \cos 0.93375115 - (-0.0013962634) \sin 0.93375115 \\ + 0.0026179939 \sin 1.09083078 \sin 0.93375115 \\ \\ \cos 1.09083078 \sin 0.93375115 + (-0.0013962634) \cos 0.93375115 \\ - 0.0026179939 \sin 1.09083078 \cos 0.93375115 \\ \\ \sin 1.09083078 \\ \end{pmatrix}$$

$$= \begin{pmatrix} 0.27764770 \\ 0.36896797 \\ 0.88701083 \end{pmatrix}$$

From equations (4-13) and (4-14),

$$\tan j = \frac{M}{L} = \frac{0.36896797}{0.27764770} = 1.32890699$$

$\boldsymbol{j}$ = 0.92569834 (radian) = **53.0386°**

$\sin q$ = 0.88701083

$\boldsymbol{q}$ **=** 1.09083078 (radian) = **62.5000°**

5.3.5.2 True Telescope Coordinates → Apparent Telescope Coordinates

Find apparent telescope coordinates from true telescope coordinates

(1) Data

Mount errors are the same as 5.3.4.2.

$D = 0.15^o = 0.15 / 180 \times p = 0.0026179939$ (radian)

$D' = -0.08^o = -0.08 / 180 \times p = -0.0013962634$ (radian)

$D'' = 0.2^o = 0.2 / 180 \times p = 0.0034906585$ (radian)

True telescope coordinates of a celestial object, $(q,\ j)$ is,

$j = 0.92569835$ (radian) $= 53.0386^o$

$q = 1.09081440$ (radian) $= 62.4991^o$

(2) Calculation

From equation (5.3-2),

$$\begin{pmatrix} \cos(q'+\Delta'')\cos j' \\ \cos(q'+\Delta'')\sin j' \\ \sin(q'+\Delta'') \end{pmatrix}$$

$$= \begin{pmatrix} (\cos 1.09081440 \cos 0.92569835 + \sin 0.92569835 \cos 0.0026179939 \sin(-0.0013962634) \\ -\sin 1.09081440 \sin 0.92569835 \sin 0.0026179939 \cos(-0.0013962634)) / \cos(-0.0013962634) \\ \\ (\cos 1.09081440 \sin 0.92569835 - \cos 0.92569835 \cos 0.0026179939 \sin(-0.0013962634) \\ +\sin 1.09081440 \cos 0.92569835 \sin 0.0026179939 \cos(-0.0013962634)) / \cos(-0.0013962634) \\ \\ (\sin 1.09081440 - \sin 0.0026179939 \sin(-0.0013962634)) \\ / \cos 0.0026179939 \cos(-0.0013962634) \end{pmatrix}$$

$$= \begin{pmatrix} 0.27467653 \\ 0.37120378 \\ 0.88701083 \end{pmatrix}$$

From equations (4-13) and (4-14),

$$\tan j' = \frac{M'}{L'} = \frac{0.37120378}{0.27467653} = 1.35142154$$

$\boldsymbol{j'} = 0.93375083$ (radian) $= \mathbf{53.5000^o}$

$\sin(q' + D'') = 0.88701083$

$q' + D'' = 1.09083078$ (radian) = 62.5000^{o}

$\boldsymbol{q'} = 62.5000^{o} - 0.2^{o} = \mathbf{62.3000^{o}}$

Approximate solution is obtained from equation (5.3-5).

$$\begin{pmatrix} \cos(q'+\Delta'')\cos j' \\ \cos(q'+\Delta'')\sin j' \\ \sin(q'+\Delta'') \end{pmatrix}$$

$$= \begin{pmatrix} \cos 1.09081440 \cos 0.92569835 + (-0.0013962634)\sin 0.92569835 \\ \qquad - (0.0026179939)\sin 1.09081440 \sin 0.92569835 \\ \cos 1.09081440 \sin 0.92569835 - (-0.0013962634)\cos 0.92569835 \\ \qquad + (0.0026179939)\sin 1.09081440 \cos 0.92569835 \\ \sin 1.09081440 \end{pmatrix}$$

$$= \begin{pmatrix} 0.27467625 \\ 0.37120343 \\ 0.88700327 \end{pmatrix}$$

From equations (4-13) and (4-14),

$$\tan j' = \frac{M'}{L'} = \frac{0.37120343}{0.27467625} = 1.35142165$$

$\boldsymbol{j'} = 0.93375087$ (radian) = $\mathbf{53.5000^{o}}$

$\sin(\boldsymbol{q'} + \boldsymbol{D''}) = 0.88700327$

$\boldsymbol{q'} + \boldsymbol{D''} = 1.09081440$ (radian) = $\mathbf{62.4991^{o}}$

$\boldsymbol{q'} = 62.5000^{o} - 0.2^{o} = \mathbf{62.2991^{o}}$

5.4 Equations for Pointing Telescope

5.4.1 Introduction

Using setting circles in telescope mount, you can point a telescope to a target object whose equatorial coordinates is known. You don't need align the telescope mount. You just have to point your telescope to two reference stars and measure the setting circle readings of the stars. Input the data to your computer, and the computer will create transformation equations. After that, you just input equatorial coordinates of a target into the computer and the computer will return the setting circle numbers for the target (ref. [6]).

The telescope coordinate system is defined as shown in figure 5.4-1. The position of a star will be specified in horizontal angle, j and elevation, q. Note that the horizontal angle is measured from right to left. This is the opposite direction to azimuth. The telescope is not necessarily leveled or aligned with any directions. Equatorial mounts and altazimuth mounts are the special cases. For equatorial mounts, j corresponds to right ascension, a and q corresponds to declination, d. For altazimuth mounts, j corresponds to –(azimuth angle) and q corresponds to altitude.

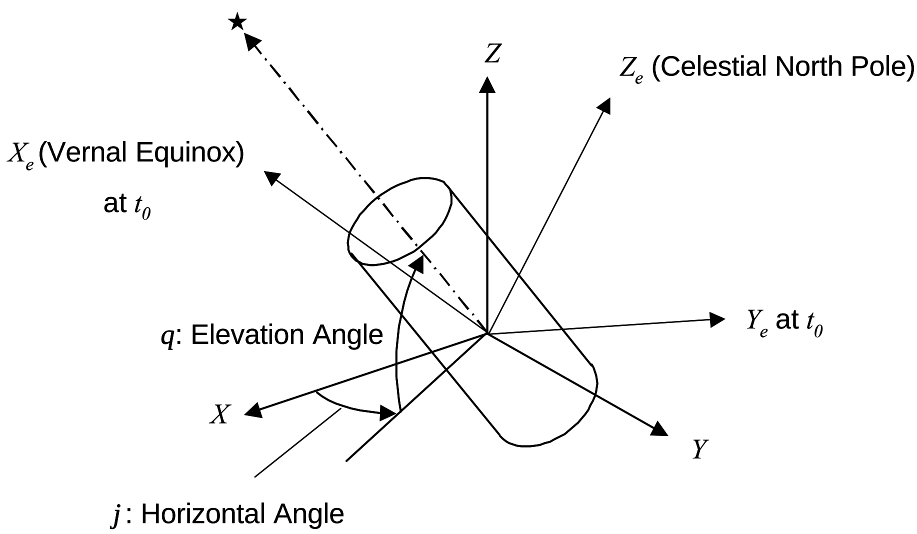


Figure 5.4-1 Telescope Coordinates and Equatorial Coordinates

5.4.2 Transformation Matrix

The relationship between telescope coordinates and equatorial coordinates is derived in this section.

Transformation from equatorial coordinates to telescope coordinates is expressed in matrix form as follows.

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{23} & T_{33} \end{bmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix} = [T] \begin{pmatrix} L \\ M \\ N \end{pmatrix} \quad \text{.... Equation (5.4-1)}$$

Transformation from telescope coordinates to equatorial coordinates is expressed as follows. This is the inverse form of equation (5.4-1).

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = [T]^{-1} \begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad \text{.... Equation (5.4-2)}$$

Where,

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \cos q \cos j \\ \cos q \sin j \\ \sin q \end{pmatrix} \quad \text{.... Equation (5.4-3)}$$

: Direction cosine of an object in telescope coordinate system

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \cos d \cos(a - k(t - t_0)) \\ \cos d \sin(a - k(t - t_0)) \\ \sin d \end{pmatrix} \quad \text{…. Equation (5.4-4)}$$

: Direction cosine of an object in equatorial coordinate system

$[T]$, $[T]^{-1}$: Transformation matrix and its inverse matrix

t : Time

t_0 : Initial time

j : Horizontal angle of an object

q : Elevation angle of an object

a : Right Ascension of an object

d : Declination of an object

$k = 1.002737908$

5.4.3 Derivation of Transformation Matrix

Suppose that data set of equatorial coordinates and telescope coordinates for three reference stars are obtained as follows.

Reference Star	Observation Time	Equatorial Coordinates		Telescope Coordinates	
		Right Ascension	Declination	Horizontal Angle	Elevation Angle
Star 1	t_1	a_1	d_1	j_1	q_1
Star 2	t_2	a_2	d_2	j_2	q_2
Star 3	t_3	a_3	d_3	j_3	q_3

Using the data above, direction cosine of each star is expressed in both telescope coordinates and equatorial coordinates.

$$\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \cos q_1 \cos j_1 \\ \cos q_1 \sin j_1 \\ \sin q_1 \end{pmatrix}$$ .... Equation (5.4-5)

$$\begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix} = \begin{pmatrix} \cos d_1 \cos(a_1 - k(t_1 - t_0)) \\ \cos d_1 \sin(a_1 - k(t_1 - t_0)) \\ \sin d_1 \end{pmatrix}$$ …. Equation (5.4-6)

$$\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} \cos q_2 \cos j_2 \\ \cos q_2 \sin j_2 \\ \sin q_2 \end{pmatrix}$$ …. Equation (5.4-7)

$$\begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix} = \begin{pmatrix} \cos d_2 \cos(a_2 - k(t_2 - t_0)) \\ \cos d_2 \sin(a_2 - k(t_2 - t_0)) \\ \sin d_2 \end{pmatrix}$$ …. Equation (5.4-8)

$$\begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_3 \cos\varphi_3 \\ \cos\theta_3 \sin\varphi_3 \\ \sin\theta_3 \end{pmatrix}$$ …. Equation (5.4-9)

$$\begin{pmatrix} L_3 \\ M_3 \\ N_3 \end{pmatrix} = \begin{pmatrix} \cos\delta_3 \cos(\alpha_3 - k(t_3 - t_0)) \\ \cos\delta_3 \sin(\alpha_3 - k(t_3 - t_0)) \\ \sin\delta_3 \end{pmatrix}$$ …. Equation (5.4-10)

Relationship between telescope coordinates and equatorial coordinates are,

$$\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = [T] \begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix}$$

$$\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = [T] \begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix}$$

$$\begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = [T] \begin{pmatrix} L_3 \\ M_3 \\ N_3 \end{pmatrix}$$

Combining the three equations above, we obtain the following equation.

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = [T] \begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix}$$

Multiplying $\begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix}^{-1}$ to the both side of the equation above, the transformation matrix is derived as follows.

$$[T] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix}^{-1}$$ …. Equation (5.4-11)

Although we use three reference stars in equation (5.4-11), two stars are enough. An independent vector (direction cosine) is created from reference star 1 and star 2 using vector product. The new direction cosines will replace direction cosines for reference star 3.

The definition of vector product is shown in figure 5.4-2 and equation (5.4-12).

$$\overrightarrow{OP_3} = \overrightarrow{OP_1} \times \overrightarrow{OP_2} = \begin{pmatrix} m_1 n_2 - n_1 m_2 \\ n_1 l_2 - l_1 n_2 \\ l_1 m_2 - m_1 l_2 \end{pmatrix}$$ …. Equation (5.4-12)

Where,

$$\overrightarrow{OP_1} = \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}, \quad \overrightarrow{OP_2} = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$$

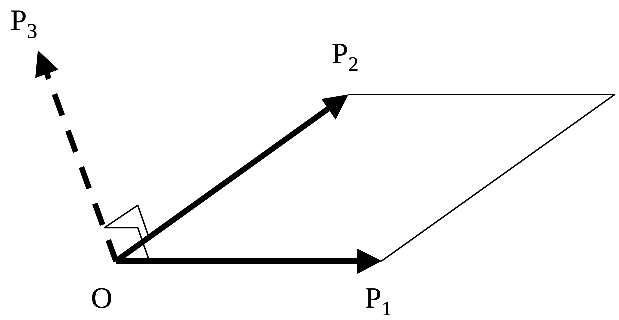


Figure 5.4-2 Vector Product

New direction cosines are created from the coordinates of the reference star 1 and the reference star 2 using equation (5.4.12). Note that the vector products are divided by the length of the vector because direction cosines should be unit length.

$$\begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = \frac{1}{\sqrt{(m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - m_1 l_2)^2}} \times \begin{pmatrix} m_1 n_2 - n_1 m_2 \\ n_1 l_2 - l_1 n_2 \\ l_1 m_2 - m_1 l_2 \end{pmatrix}$$

…. Equation (5.4-13)

$$\begin{pmatrix} L_3 \\ M_3 \\ N_3 \end{pmatrix} = \frac{1}{\sqrt{(M_1N_2 - N_1M_2)^2 + (N_1L_2 - L_1N_2)^2 + (L_1M_2 - M_1L_2)^2}} \times \begin{pmatrix} M_1N_2 - N_1M_2 \\ N_1L_2 - L_1N_2 \\ L_1M_2 - M_1L_2 \end{pmatrix}$$

…. Equation (5.4-14)

Use equations (5.4-13) and (5.4-14) in equation (5.4-11) instead of (5.4-9) and (5.4-10).

5.4.4 Example Calculation

The following data was measured using my 12.5 inch Dobsonian with setting circles. Calculate the transformation matrix from telescope coordinates and equatorial coordinates assuming the mount does not have fabrication errors.

Reference Star	Observation Time	Equatorial Coordinates		Telescope Coordinates	
		Right Ascension	Declination	Horizontal Angle	Elevation Angle
Initial Time	t_0 = 21h00m00s = 5.497787	--	--	--	--
Star 1: α And	t_1 = 21h27m56s = 5.619669	a_1 = 0h07m54s = 0.034470	d_1 = 29.038° = 0.506809	j_1 = 99.25° = 1.732239	q_1 = = 83.87° = 1.463808
Star 2: α Umi	t_2 = 21h37m02s = 5.659376	a_2 = 2h21m45s = 0.618501	d_2 = 89.222° = 1.557218	j_2 = 310.98° = 5.427625	q_2 = 35.04° = 0.611563

From equation (5.4-5),

$$\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \cos 1.463808 \cos 1.732239 \\ \cos 1.463808 \sin 1.732239 \\ \sin 1.463808 \end{pmatrix} = \begin{pmatrix} -0.0171648 \\ 0.105396 \\ 0.994282 \end{pmatrix}$$

From equation (5.4-6),

$$\begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix} = \begin{pmatrix} \cos 0.506809 \cos(0.034470 - 1.002737908(5.619669 - 5.497787)) \\ \cos 0.506809 \sin(0.034470 - 1.002737908(5.619669 - 5.497787)) \\ \sin 0.506809 \end{pmatrix}$$

$$= \begin{pmatrix} 0.870934 \\ -0.0766175 \\ 0.485390 \end{pmatrix}$$

From equation (5.4-7),

$$\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} \cos 0.611563 \cos 5.427625 \\ \cos 0.611563 \sin 5.427625 \\ \sin 0.611563 \end{pmatrix} = \begin{pmatrix} 0.536934 \\ -0.618107 \\ 0.574148 \end{pmatrix}$$

From equation (5.4-8),

$$\begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix} = \begin{pmatrix} \cos 1.557218 \cos(0.618501 - 1.002737908(5.659376 - 5.497787)) \\ \cos 1.557218 \sin(0.618501 - 1.002737908(5.659376 - 5.497787)) \\ \sin 1.557218 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0121877 \\ 0.00598490 \\ 0.999908 \end{pmatrix}$$

From equation (5.4-13),

$$\begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = \frac{1}{\sqrt{\begin{array}{r} (0.105396\times0.574148-0.994282\times(-0.618107))^2 \\ +(0.994282\times0.536934-(-0.0171648)\times0.574148)^2 \\ +((-0.0171648)(-0.618107)-0.105396\times0.536934)^2 \end{array}}}$$

$$\times\begin{pmatrix} 0.105396\times0.574148-0.994282\times(-0.618107) \\ 0.994282\times0.536934-(-0.0171648)\times0.574148 \\ (-0.0171648)\times(-0.618107)-0.105396\times(0.536934) \end{pmatrix}$$

$$=\begin{pmatrix} 0.777717 \\ 0.626379 \\ -0.0529714 \end{pmatrix}$$

From equation (5.4-14),

$$\begin{pmatrix} L_3 \\ M_3 \\ N_3 \end{pmatrix} = \frac{1}{\sqrt{\begin{array}{r} ((-0.0766175)\times0.999908-0.485390\times0.00598490)^2 \\ +(0.485390\times0.0121877-0.870934\times0.999908)^2 \\ +(0.870934\times0.00598490-(-0.0766175)\times0.0121877)^2 \end{array}}}$$

$$\times\begin{pmatrix} (-0.0766175)\times0.999908-0.485390\times0.00598490 \\ 0.485390\times0.0121877-0.870934\times0.999908 \\ 0.870934\times0.00598490-(-0.0766175)\times0.0121877 \end{pmatrix}$$

$$=\begin{pmatrix} -0.0915436 \\ -0.995776 \\ 0.00707598 \end{pmatrix}$$

The inverse matrix is,

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix}^{-1} = \begin{bmatrix} 0.870934 & 0.0121877 & -0.0915436 \\ -0.0766175 & 0.00598490 & -0.995776 \\ 0.485390 & 0.999908 & 0.00707598 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1.146349 & -0.105481 & -0.0133413 \\ -0.555830 & 0.0582508 & 1.006518 \\ -0.0915436 & -0.995776 & 0.00707598 \end{bmatrix}$$

From equation (5.4-11), we obtain the following transform matrix.

$$[T] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -0.0171648 & 0.536934 & 0.777717 \\ 0.105396 & -0.618107 & 0.626379 \\ 0.994282 & 0.574148 & -0.0529714 \end{bmatrix} \begin{bmatrix} 1.146349 & -0.105481 & -0.0133413 \\ -0.555830 & 0.0592508 & 1.006518 \\ -0.0915436 & -0.995776 & 0.00707598 \end{bmatrix}$$

$$= \begin{bmatrix} -0.38932 & -0.74134 & 0.54617 \\ 0.40704 & -0.67086 & -0.61911 \\ 0.82552 & -0.018686 & 0.56425 \end{bmatrix}$$

If you want to aim the telescope at *b* Cet (*a* = 0h43m07s, *d* = -18.038°) at 21h52m12s, from equation (5.4-1),

$$a = 0h43m07s = 0.188132\, radian$$

$$d = -18.038^\circ = -0.314822\, radian$$

$$t = 21h52m12s = 5.725553\, radian$$

From equation (5.4-1),

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \cos d \cos(a - k(t - t_0)) \\ \cos d \sin(a - k(t - t_0)) \\ \sin d \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-0.314822) \times \cos(0.188132 - 1.002737908 \times (5.725553 - 5.497787)) \\ \cos(-0.314822) \times \sin(0.188132 - 1.002737908 \times (5.725553 - 5.497787)) \\ \sin(-0.314822) \end{pmatrix}$$

$$= \begin{pmatrix} 0.950081 \\ -0.0382687 \\ -0.309647 \end{pmatrix}$$

From equation (5.4-1),

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = [T] \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$

$$= \begin{bmatrix} -0.38932 & -0.74134 & 0.54617 \\ 0.40704 & -0.67086 & -0.61911 \\ 0.82552 & -0.018686 & 0.56425 \end{bmatrix} \begin{pmatrix} 0.950081 \\ -0.0382687 \\ -0.309647 \end{pmatrix}$$

$$= \begin{pmatrix} -0.510635 \\ 0.604099 \\ 0.610308 \end{pmatrix}$$

From equations (4-13) and (4-14), telescope coordinates are calculated as follows.

$$\tan j = \frac{0.604099}{-0.510635} = -1.183035$$

$$j = 2.272546\, radian = 130.21^\circ$$

$$\sin q = 0.610308$$

$$q = 0.656449\, radian = 37.61^\circ$$

This calculated telescope coordinates is very close to the measured telescope coordinates, j = 130.46°, q = 37.67°.

5.5 Polar Axis Misalignment Determination

The matrix method is applied to derive the equations of the declination drift method for polar axis misalignment determination in this section.

The declination drift method was proposed by Challis [5] to determine the polar axis misalignment of equatorial mount. The advantage of the declination drift method is its simplicity of the measurement.

5.5.1 Derivation of Equations

5.5.1.1 Relationship between Coordinate Systems

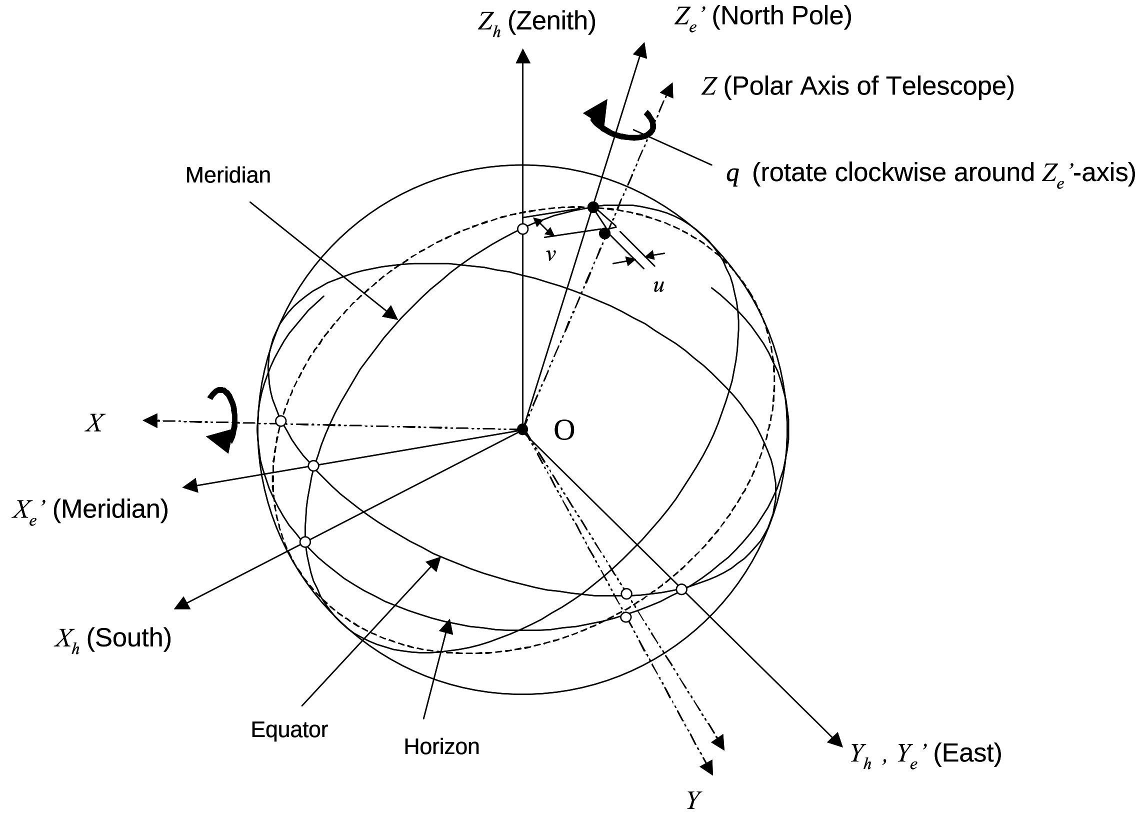
Following Coordinate systems are used. See figure 5.5-1.

Equatorial coordinate system is X_e'-Y_e'-Z_e'. Z_e'-axis directs to the north pole. Y_e'-axis is in the horizontal plane and directs to the east. X_e'-axis is on the meridian. Polar coordinates of a celestial object in the equatorial coordinate system is $(-H, d)$, where H is hour angle and d is declination.

Telescope coordinate system is X-Y-Z and its polar coordinates is (x, z).

Misalignment of the telescope polar axis Z from the celestial polar axis Z_e' is defined as follows.

First, equatorial coordinate system X_e'-Y_e'-Z_e' is rotated q clockwise around Z_e'-axis (polar axis), then the new coordinate system is rotated g clockwise around the new X-axis.



<u>**Figure 5.5-1 Misalignment of Telescope Polar Axis**</u>

Direction cosine of a star in equatorial coordinates is,

$$\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos\boldsymbol{d}\,\cos(-H) \\ \cos\boldsymbol{d}\,\sin(-H) \\ \sin\boldsymbol{d} \end{pmatrix}$$ … Equation (5.5.1-1)

Direction cosine of the same star in telescope coordinates is,

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \cos V \cos x \\ \cos V \sin x \\ \sin V \end{pmatrix}$$ … Equation (5.5.1-2)

Relationship between the coordinate systems is derived as follows using equations shown in section 4.2.

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos g & -\sin g \\ 0 & \sin g & \cos g \end{bmatrix} \begin{bmatrix} \cos q & -\sin q & 0 \\ \sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix}$$

$$= \begin{bmatrix} \cos q & -\sin q & 0 \\ \cos g \sin q & \cos g \cos q & -\sin g \\ \sin g \sin q & \sin g \cos q & \cos g \end{bmatrix} \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix}$$

… Equation (5.5.1-3)

$$\begin{pmatrix} \cos V \cos x \\ \cos V \sin x \\ \sin V \end{pmatrix} = \begin{bmatrix} \cos q & -\sin q & 0 \\ \cos g \sin q & \cos g \cos q & -\sin g \\ \sin g \sin q & \sin g \cos q & \cos g \end{bmatrix} \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix}$$

… Equation (5.5.1-4)

5.5.1.2 Basic Equations for Declination Drift Method

Two stars are selected for measurement. Point the telescope to the first star and drive the equatorial mount around polar axis only. Measure the drift of the star in declination in a certain time interval. Same measurement is done for the second star.

Assume that declination drifts of two stars are obtained from observation as shown in table 5.5.1-1. Atmospheric refraction is neglected in this section. Effect of atmospheric refraction will be discussed in section 5.5.1.4.

Table 5.5.1-1 Observed Data

Star	Position of Star		Time		Drift of Declination (Refraction is neglected.)
	Right Ascension, a	Declination, d	Start	End	
Star 1	a_1	d_1	t_{1a}	t_{1b}	z_{1b} - z_{1a}
Star 2	a_2	d_2	t_{2a}	t_{2b}	z_{2b} - z_{2a}

From equation (5.5.1-4),

$$\sin V = \sin g \sin q \cos d \cos(-H) + \sin g \cos q \cos d \sin(-H) + \cos g \sin d$$

… Equation (5.5.1-5)

Using the data from star 1 in table 5.5.1-1,

$$\sin V_{1a} = \sin g \, \sin q \, \cos d_1 \, \cos(-H_{1a}) + \sin g \, \cos q \, \cos d_1 \, \sin(-H_{1a}) + \cos g \, \sin d_1$$

… Equation (5.5.1-6)

Where,

H_{1a}: Hour angle of the first star at time t_{1a}

H_{1b}: Hour angle of the first star at time t_{1b}

z_{1a}: Declination in Telescope Coordinates at time t_{1a}

d_1: Declination of the first star in Equatorial Coordinates

Assuming that g is very small, equation (5.5.1-6) is expressed as,

$$\sin V_{1a} \cong g \, \sin q \, \cos d_1 \, \cos(-H_{1a}) + g \, \cos q \, \cos d_1 \, \sin(-H_{1a}) + \sin d_1$$
$$= \sin d_1 + g \, \cos d_1 (\sin q \, \cos(-H_{1a}) + \cos q \, \sin(-H_{1a}))$$

… Equation (5.5.1-7)

Considering the following relationship (assuming that Δ is very small),

$$\sin(d_1 + \Delta) = \sin d_1 \cos \Delta + \cos d_1 \sin \Delta \cong \sin d_1 + \Delta \cos d_1$$

We get the following equation from equation (5.5.1-7).

$$V_{1a} \cong d_1 + g (\sin q \, \cos(-H_{1a}) + \cos q \, \sin(-H_{1a}))$$
$$= d_1 + g \, \sin q \, \cos(-H_{1a}) + g \, \cos q \, \sin(-H_{1a})$$

… Equation (5.5.1-8)

Same equation is derived for time t_{1b}.

$$V_{1b} \cong d_1 + g \, \sin q \, \cos(-H_{1b}) + g \, \cos q \, \sin(-H_{1b})$$

… Equation (5.5.1-9)

Retracting equation (5.5.1-8) from (5.5.1-9),

$$V_{1b} - V_{1a} = g \, \sin q (\cos(-H_{1b}) - \cos(-H_{1a})) + g \, \cos q (\sin(-H_{1b}) - \sin(-H_{1a}))$$

… Equation (5.5.1-10)

Putting $u = g \sin q$, $v = g \cos q$ in equation (5.5.1-10),

$$V_{1b} - V_{1a} = u(\cos(-H_{1b}) - \cos(-H_{1a})) + v(\sin(-H_{1b}) - \sin(-H_{1a}))$$

… Equation (5.5.1-11)

Using the data of star 2 in table 5.5.1-1,

$$V_{2b} - V_{2a} = u(\cos(-H_{2b}) - \cos(-H_{2a})) + v(\sin(-H_{2b}) - \sin(-H_{2a}))$$

… Equation (5.5.1-12)

Equations (5.5.1-11) and (5.5.1-12) are the basic equations of declination drift method.

When you obtain the data in table 5.5.1-1 from observation, you can calculate the polar axis misalignment u and v from equations (5.5.1-11) and (5.5.1-12).

Two important comments can be made based on the equations.

(1) In order to maximize the accuracy of the method, it is desirable to take two stars which locations at observation are nearly 90 degree apart.

(2) It is not necessary to select stars near equator. Stars far from celestial equator can work. This conclusion is derived from the fact that the declinations of the stars do not appear in equations (5.5.1-11) and (5.5.1-12).

5.5.1.3 Challis' Method

Challis' original declination drift method requires three measurements with one star. See table 5.5.1-2 for the required data.

Table 5.5.1-2 Observed Data – Challis' Method

Star	Position of Star		Time		Drift of Declination (Refraction is neglected.)
	Right Ascension, a	Declination, d	Start	End	
Star 1	a_1	d_1	t_a	t_b	z_b - z_a
			t_a	t_c	z_c - z_a

Based on equations (5.5.1-11) and (5.5.1-12),

$$V_b - V_a = u(\cos(-H_b) - \cos(-H_a)) + v(\sin(-H_b) - \sin(-H_a))$$

… Equation (5.5.1-13)

$$V_c - V_a = u(\cos(-H_c) - \cos(-H_a)) + v(\sin(-H_c) - \sin(-H_a))$$

… Equation (5.5.1-14)

5.5.1.4 Compensation of Atmospheric Refraction

Effect of atmospheric refraction is included in the measured data.

Table 5.5.1-3 Observed Data

Star	Position of Star		Time		Drift of Declination (Refraction is included.)
	Right Ascension, a	Declination, d	Start	End	
Star 1	a_1	d_1	t_{1a}	t_{1b}	$z_{1b}' - z_{1a}'$
Star 2	a_2	d_2	t_{2a}	t_{2b}	$z_{2b}' - z_{2a}'$

Altitude of the star at observed instant is necessary to calculate the atmospheric refraction. Relationship between equatorial coordinates and altazimuth coordinates is (see section 5.1.1),

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{bmatrix} \sin f & 0 & -\cos f \\ 0 & 1 & 0 \\ \cos f & 0 & \sin f \end{bmatrix} \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} \quad \text{… Equation (5.5.1-15)}$$

Where,

f is observer's latitude.

$$\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} = \begin{pmatrix} \cos h \cos(-A) \\ \cos h \sin(-A) \\ \sin h \end{pmatrix} \quad \text{… Equation (5.5.1-16)}$$

A is azimuth measured westward from the South and h is "airless" altitude.

$$L_h = \cos h \cos(-A) = \sin f \cos d \cos(-H) - \cos f \sin d$$

$$M_h = \cos h \sin(-A) = \cos d \sin(-H)$$

$$N_h = \sin h = \cos f \cos d \cos(-H) + \sin f \sin d$$

$$\tan(-A) = \frac{M_h}{L_h} \quad \text{… Equation (5.5.1-17)}$$

From equation (15.2) in ref. [2] (page 101), atmospheric refraction R is expressed as follows. R is added to "airless" altitude h to obtain apparent altitude. Note that equation (5.5.1-18) is valid for the altitude larger than 15 degree.

$$R=\frac{58.276}{3600\times180/p}\tan\left(\frac{p}{2}-h\right)-\frac{0.0824}{3600\times180/p}\tan^3\left(\frac{p}{2}-h\right)$$ (in radian)

… Equation (5.5.1-18)

Using the following relationship,

$$\tan\left(\frac{p}{2}-h\right)=\frac{\cos h}{\sin h}=\frac{\sqrt{1-\sin^2 h}}{\sin h}=\frac{\sqrt{1-N_h{}^2}}{N_h}$$ … Equation (5.5.1-19)

Equation (5.5.1-18) is expressed as follows.

$$R=\frac{58.276}{3600\times180/p}\frac{\sqrt{1-N_h{}^2}}{N_h}-\frac{0.0824}{3600\times180/p}\left(\frac{\sqrt{1-N_h{}^2}}{N_h}\right)^3$$

… Equation (5.5.1-20)

Apparent altitude h_0 is expressed as follows.

$h_0=h+R$ … Equation (5.5.1-21)

Refracted position of the star in altazimuth coordinates $\begin{pmatrix} L_h' \\ M_h' \\ N_h' \end{pmatrix}$ is,

$$\begin{pmatrix} L_h' \\ M_h' \\ N_h' \end{pmatrix}=\begin{pmatrix} \cos(h+R)\cos(-A) \\ \cos(h+R)\sin(-A) \\ \sin(h+R) \end{pmatrix}=\begin{pmatrix} \cos h\cos R\cos(-A)-\sin h\sin R\cos(-A) \\ \cos h\cos R\sin(-A)-\sin h\sin R\sin(-A) \\ \sin h\cos R+\cos h\sin R \end{pmatrix}$$
$$\approx\begin{pmatrix} L_h-R\sin h\cos(-A) \\ M_h-R\sin h\sin(-A) \\ N_h+R\cos h \end{pmatrix}=\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix}+R\begin{pmatrix} -\sin h\cos A \\ \sin h\sin A \\ \cos h \end{pmatrix}$$

… Equation (5.5.1-22)

Then, refracted position of the star in equatorial coordinates $\begin{pmatrix} L_e'' \\ M_e'' \\ N_e'' \end{pmatrix}$ is,

$$\begin{pmatrix} L_e'' \\ M_e'' \\ N_e'' \end{pmatrix} = \begin{bmatrix} \sin f & 0 & \cos f \\ 0 & 1 & 0 \\ -\cos f & 0 & \sin f \end{bmatrix} \begin{pmatrix} L_h' \\ M_h' \\ N_h' \end{pmatrix}$$

$$= \begin{bmatrix} \sin f & 0 & \cos f \\ 0 & 1 & 0 \\ -\cos f & 0 & \sin f \end{bmatrix} \left(\begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} + R \begin{pmatrix} -\sin h \cos A \\ \sin h \sin A \\ \cos h \end{pmatrix} \right)$$

$$= \begin{bmatrix} \sin f & 0 & \cos f \\ 0 & 1 & 0 \\ -\cos f & 0 & \sin f \end{bmatrix} \begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix} + R \begin{bmatrix} \sin f & 0 & \cos f \\ 0 & 1 & 0 \\ -\cos f & 0 & \sin f \end{bmatrix} \begin{pmatrix} -\sin h \cos A \\ \sin h \sin A \\ \cos h \end{pmatrix}$$

$$= \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} + R \begin{pmatrix} -\sin f \sin h \cos A + \cos f \cos h \\ \sin h \sin A \\ \cos f \sin h \cos A + \sin f \cos h \end{pmatrix}$$

… Equation (5.5.1-23)

Refracted position of the star in telescope coordinates $\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix}$ is derived from equations (5.5.1-3) and (5.5.1-23).

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = \begin{bmatrix} \cos q & -\sin q & 0 \\ \cos g \sin q & \cos g \cos q & -\sin g \\ \sin g \sin q & \sin g \cos q & \cos g \end{bmatrix} \left(\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} + R \begin{pmatrix} -\sin f \sin h \cos A + \cos f \cos h \\ \sin h \sin A \\ \cos f \sin h \cos A + \sin f \cos h \end{pmatrix} \right)$$

… Equation (5.5.1-24)

Putting observed data of star 1 at t_{1b} in table 5.5.1-3 to equation (5.5.1-23),

$$\begin{aligned} \sin V_{1b}' = & \sin g \sin q \left(\cos d_1 \cos(-H_{1b}) - R_{1b} \sin f \sin h_{1b} \cos A_{1b} + R_{1b} \cos f \cos h_{1b} \right) \\ & + \sin g \cos q \left(\cos d_1 \sin(-H_{1b}) + R_{1b} \sin h_{1b} \sin A_{1b} \right) \\ & + \cos g \left(\sin d_1 + R_{1b} \cos f \sin h_{1b} \cos A_{1b} + R_{1b} \sin f \cos h_{1b} \right) \end{aligned}$$

Considering g and R_{1b} are very small,

$$\sin V_{1b}' \cong g \sin q \cos d_1 \cos(-H_{1b}) + g \cos q \cos d_1 \sin(-H_{1b}) + \sin d_1$$
$$+ R_{1b} \cos f \sin h_{1b} \cos A_{1b} + R_{1b} \sin f \cos h_{1b}$$
$$= \sin d_1 + \left(g \sin q \cos(-H_{1b}) + g \cos q \sin(-H_{1b}) + R_{1b} \frac{\cos f \sin h_{b1} \cos A_{1b} + \sin f \cos h_{1b}}{\cos d_1} \right) \cos d_1$$

Considering the following relationship (assuming that Δ is very small),

$$\sin(d_1 + \Delta) = \sin d_1 \cos\Delta + \cos d_1 \sin \Delta \cong \sin d_1 + \Delta \cos d_1$$

$$V_{1b}' = d_1 + g \sin q \cos(-H_{1b}) + g \cos q \sin(-H_{1b}) + R_{1b} \frac{\cos f \sin h_{1b} \cos A_{1b} + \sin f \cos h_{1b}}{\cos d_1}$$

… Equation (5.5.1-25)

Same equation is derived for data of star 1 at t_{1a} in table 5.5.1-3.

$$V_{1a}' = d_1 + g \sin q \cos(-H_{1a}) + g \cos q \sin(-H_{1a}) + R_{1a} \frac{\cos f \sin h_{1a} \cos A_{1a} + \sin f \cos h_{1a}}{\cos d_1}$$

… Equation (5.5.1-26)

Retracting equation (5.5.1-25) from (5.5.1-26),

$$V_{1b}' - V_{1a}' = Ag \sin q + Bg \cos q + C$$ … Equation (5.5.1-27)

Where,

$$A = \cos(-H_{1b}) - \cos(-H_{1a})$$
$$B = \sin(-H_{1b}) - \sin(-H_{1a})$$
$$C = R_{1b} \frac{\cos f \sin h_{1b} \cos A_{1b} + \sin f \cos h_{1b}}{\cos d_1} - R_{1a} \frac{\cos f \sin h_{1a} \cos A_{1a} + \sin f \cos h_{1a}}{\cos d_1}$$
$$= R_{1b} \frac{L_{h1b} \cos f \tan h_{1b} + \sin f \cos h_{1b}}{\cos d_1} - R_{1a} \frac{L_{h1a} \cos f \tan h_{1a} + \sin f \cos h_{1a}}{\cos d_1}$$

Same equation is derived for data of star 2.

$$V_{2b}' - V_{2a}' = Dg \sin q + Eg \cos q + F$$ … Equation (5.5.1-28)

Where,

$$D = \cos(-H_{2b}) - \cos(-H_{2a})$$
$$E = \sin(-H_{2b}) - \sin(-H_{2a})$$

$$F = R_{2b}\,\frac{\cos f\,\sin h_{2b}\,\cos A_{2b} + \sin f\,\cos h_{2b}}{\cos \mathbf{d}_2} - R_{2a}\,\frac{\cos f\,\sin h_{2a}\,\cos A_{2a} + \sin f\,\cos h_{2a}}{\cos \mathbf{d}_2}$$

$$= R_{2b}\,\frac{L_{h2b}\,\cos f\,\tan h_{2b} + \sin f\,\cos h_{2b}}{\cos \mathbf{d}_2} - R_{2a}\,\frac{L_{h2a}\,\cos f\,\tan h_{2a} + \sin f\,\cos h_{2a}}{\cos \mathbf{d}_2}$$

Putting $u = g\,\sin q$, $v = g\,\cos q$ in equations (5.5.1-23) and (5.5.1-24),

$Au + Bv + C - (V_{1b}' - V_{1a}') = 0$ … Equation (5.5.1-29)

$Du + Ev + F - (V_{2b}' - V_{2a}') = 0$ … Equation (5.5.1-30)

Equations (5.5.1-29), (5.5.1-30), (5.5.1-17) and (5.5.1-20) are the equations for declination drift method with atmospheric refraction.

5.5.2 Example Calculations

5.5.2.1 Two Star Declination Drift Method with Atmospheric Refraction Neglected

(1) Observed Data

Observed data is shown in table 5.5.2-1.

Table 5.5.2-1 Observed Data

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 14h15m49s	d_1 = 19°10'29"	t_{1a} = 2001 May 24, 21h00m00s	t_{1b} = 2001 May 24, 21h50m00s	$z_{1b} - z_{1a}$ = -34.52"
α Boo	a_2 = 14h15m49s	d_2 = 19°10'29"	t_{2a} = 2001 May 24, 21h50m00s	t_{2b} = 2001 May 24, 22h23m00s	$z_{2b} - z_{2a}$ = -65.88"
Observation Location: Latitude, f = 52°09'20".32N, Longitude, L = 0°0'38".36E					

All the data is converted to radian.

Table 5.5.2-2 Observed Data (in radian)

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 3.73420466	d_1 = 0.33466204	t_{1a} = 2001 May 24, 5.49778714	t_{1b} = 2001 May 24, 5.71595330	$z_{1b} - z_{1a}$ = -0.00016736
α Boo	a_2 = 3.73420466	d_2 = 0.33466204	t_{2a} = 2001 May 24, 5.71595330	t_{2b} = 2001 May 24, 5.85994296	$z_{2b} - z_{2a}$ = -0.00031940
Observation Location: Latitude, f = 0.91028772, Longitude, L = -0.00018597					

(2) Sidereal Time

Julian Day number JD corresponding to 2001 May 24, 0h UT is 2452053.5 (chapter 7 in ref. [2]).

The sidereal time at Greenwich at 2001 May 24, 0h UT is (chapter 11 of ref. [2]),

$$T = \frac{JD - 2451545.0}{36525} = \frac{2452053.5 - 2451545.0}{36525} = 0.01392197125$$

$$q_0 \ at \ 0hUT = 100.46061837 + 36000.770053608T + 0.000387933T^2 - \frac{T^3}{38710000}$$

$$= 601.662304\deg = 241.662304\deg = 4.21780288 = 16h06m38.95s$$

(3) Hour Angle

From chapter 12 of ref. [2], hour angles are calculated as follows.

$$H = q_0 - L - a$$

$$H_{1a} = 4.21780288 + 1.00273790935 \times 5.49778714 - (-0.00018597) - 3.73420466$$
$$= 5.99662377$$
$$H_{1b} = 4.21780288 + 1.00273790935 \times 5.71595330 - (-0.00018597) - 3.73420466$$
$$= 6.21538725$$
$$H_{2a} = 4.21780288 + 1.00273790935 \times 5.71595330 - (-0.00018597) - 3.73420466$$
$$= 6.21538725$$
$$H_{2b} = 4.21780288 + 1.00273790935 \times 5.85994296 - (-0.00018597) - 3.73420466$$
$$= 6.35977114$$

(4) Basic Equations

From equation (5.5.1-11),

$$-0.00016736 = u(\cos(-6.21538725) - \cos(-5.99662377)) + v(\sin(-6.21538725) - \sin(-5.99662377))$$

$$-0.00016736 = 0.038481147u - 0.21490953v \quad \text{… Equation (5.5.2-1)}$$

From equation (5.5.1-12),

$$-0.00031940 = u(\cos(-6.35977114) - \cos(-6.21538725)) + v(\sin(-6.35977114) - \sin(-6.21538725))$$

$$-0.00031940 = -0.00063385367u - 0.14425712v \qquad \text{… Equation (5.5.2-2)}$$

Solving equations (5.5.2-1) and (5.5.2-2),

u = 0.007916 radian = 0.454^{o} = 1633”

v = 0.002179 radian = 0.125^{o} = 449”

5.5.2.2 Challis' Method with Atmospheric Refraction Neglected

(1) Observed Data

Observed data is shown in table 5.5.2-3.

Table 5.5.2-3 Observed Data

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 14h15m49s	d_1 = $19^o10'29''$	t_a = 2001 May 24, 21h00m00s	t_b = 2001 May 24, 21h50m00s	z_b - z_a = -34.52"
			t_a = 2001 May 24, 21h00m00s	t_c = 2001 May 24, 22h50m00s	z_c - z_a = -100.40"
Observation Location: Latitude, f = $52^o09'20''.32$N, Longitude, L = $0^o0'38''.36$E					

All the data is converted to radian.

Table 5.5.2-4 Observed Data (in radian)

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 3.73420466	d_1 = 0.33466204	t_a = 2001 May 24, 5.49778714	t_b = 2001 May 24, 5.71595330	z_b - z_a = -0.00016736
			t_a = 2001 May 24, 5.49778714	t_c = 2001 May 24, 5.85994296	z_c - z_a = -0.00048675
Observation Location: Latitude, f = 0.91028772, Longitude, L = -0.00018597					

(2) Sidereal Time

Julian Day number JD corresponding to 2001 May 24, 0h UT is 2452053.5 (chapter 7 in ref.

[2]).

The sidereal time at Greenwich at 2001 May 24, 0h UT is (chapter 11 of ref. [2]),

$$T = \frac{JD - 2451545.0}{36525} = \frac{2452053.5 - 2451545.0}{36525} = 0.01392197125$$

$$q_0 \; at \; 0hUT = 100.46061837 + 36000.770053608T + 0.000387933T^2 - \frac{T^3}{38710000}$$

$$= 601.662304 \deg = 241.662304 \deg = 4.21780288 = 16h06m38.95s$$

(3) Hour Angle

From chapter 12 of ref. [2], hour angles are calculated as follows.

$$H = q_0 - L - a$$

$$H_a = 4.21780288 + 1.00273790935 \times 5.49778714 - (-0.00018597) - 3.73420466$$
$$= 5.99662377$$
$$H_b = 4.21780288 + 1.00273790935 \times 5.71595330 - (-0.00018597) - 3.73420466$$
$$= 6.21538725$$
$$H_c = 4.21780288 + 1.00273790935 \times 5.85994296 - (-0.00018597) - 3.73420466$$
$$= 6.35977114$$

(4) Basic Equations

From equation (5.5.1-13),

$$-0.00016736 = u(\cos(-6.21538725) - \cos(-5.99662377))$$
$$+ v(\sin(-6.21538725) - \sin(-5.99662377))$$
$$-0.00016736 = 0.038481147u - 0.21490953v$$ … Equation (5.5.2-3)

From equation (5.5.1-14),

$$-0.00031940 = u(\cos(-6.35977114) - \cos(-5.99662377))$$
$$+ v(\sin(-6.35977114) - \sin(-5.99662377))$$
$$-0.00048675 = 0.037847293u - 0.3591666v$$ … Equation (5.5.2-4)

Solving equations (5.5.2-3) and (5.5.2-4),

u = 0.008051 radian = 0.447^{o} = 1661”

v = 0.002204 radian = 0.125^{o} = 455”

5.5.2.3 Two Star Drift Method with Atmospheric Refraction Compensated

(1) Observed Data

Same data as section 5.5.2.1 is used.

Table 5.5.2-5 Observed Data

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 14h15m49s	d_1 = 19°10'29"	t_{1a} = 2001 May 24, 21h00m00s	t_{1b} = 2001 May 24, 21h50m00s	$z_{1b} - z_{1a}$ = -34.52"
α Boo	a_2 = 14h15m49s	d_2 = 19°10'29"	t_{2a} = 2001 May 24, 21h50m00s	t_{2b} = 2001 May 24, 22h23m00s	$z_{2b} - z_{2a}$ = -65.88"
Observation Location: Latitude, f = 52°09'20".32N, Longitude, L = 0°0'38".36E					

All the data is converted to radian.

Table 5.5.2-6 Observed Data (in radian)

Star	Position of Star		Time		Drift of Declination
	Right Ascension, a	Declination, d	Start	End	
α Boo	a_1 = 3.73420466	d_1 = 0.33466204	t_{1a} = 2001 May 24, 5.49778714	t_{1b} = 2001 May 24, 5.71595330	$z_{1b} - z_{1a}$ = -0.00016736
α Boo	a_2 = 3.73420466	d_2 = 0.33466204	t_{2a} = 2001 May 24, 5.71595330	t_{2b} = 2001 May 24, 5.85994296	$z_{2b} - z_{2a}$ = -0.00031940
Observation Location: Latitude, f = 0.91028772, Longitude, L = -0.00018597					

(2) Sidereal Time

Julian Day number JD corresponding to 2001 May 24, 0h UT is 2452053.5 (chapter 7 in ref. [2]).

The sidereal time at Greenwich at 2001 May 24, 0h UT is (chapter 11 of ref. [2]),

$$T = \frac{JD - 2451545.0}{36525} = \frac{2452053.5 - 2451545.0}{36525} = 0.01392197125$$

$$q_0 \ at \ 0hUT = 100.46061837 + 36000.770053608T + 0.000387933T^2 - \frac{T^3}{38710000}$$
$$= 601.662304 \deg = 241.662304 \deg = 4.21780288 = 16h06m38.95s$$

(3) Hour Angle

From chapter 12 of ref. [2], hour angles are calculated as follows.

$$H = q_0 - L - a$$

$$H_{1a} = 4.21780288 + 1.00273790935 \times 5.49778714 - (-0.00018597) - 3.73420466$$
$$= 5.99662377$$
$$H_{1b} = 4.21780288 + 1.00273790935 \times 5.71595330 - (-0.00018597) - 3.73420466$$
$$= 6.21538725$$
$$H_{2a} = 4.21780288 + 1.00273790935 \times 5.71595330 - (-0.00018597) - 3.73420466$$
$$= 6.21538725$$
$$H_{2b} = 4.21780288 + 1.00273790935 \times 5.85994296 - (-0.00018597) - 3.73420466$$
$$= 6.35977114$$

(4) Basic Equations

Atmospheric refraction is calculated using equations (5.5.1-17) and (5.5.1-20).

$$L_{h1a} = \sin(0.91028772)\cos(0.33466204) * \cos(-5.99662377) - \cos(0.91028772)\sin(0.33466204)$$
$$= 0.51394425$$
$$L_{h1b} = \sin(0.91028772)\cos(0.33466204) * \cos(-6.21538725) - \cos(0.91028772)\sin(0.33466204)$$
$$= 0.54264618$$
$$L_{h2a} = \sin(0.91028772)\cos(0.33466204) * \cos(-6.21538725) - \cos(0.91028772)\sin(0.33466204)$$
$$= 0.54264618$$

$$L_{h2b} = \sin(0.91028772)\cos(0.33466204) * \cos(-6.35977114) - \cos(0.91028772)\sin(0.33466204)$$
$$= 0.54217341$$

$$N_{h1a} = \cos 0.91028772 \cos 0.33466204 \cos(-5.99662377) + \sin 0.91028773 \sin 0.3366204$$
$$= 0.81522146$$
$$N_{h1b} = \cos 0.91028772 \cos 0.33466204 \cos(-6.21538725) + \sin 0.91028773 \sin 0.3366204$$
$$= 0.83752057$$
$$N_{h2a} = \cos 0.91028772 \cos 0.33466204 \cos(-6.21538725) + \sin 0.91028773 \sin 0.3366204$$
$$= 0.83752057$$
$$N_{h2b} = \cos 0.91028772 \cos 0.33466204 \cos(-6.35977114) + \sin 0.91028773 \sin 0.3366204$$
$$= 0.83715326$$

$$h_{1a} = \sin^{-1} N_{h1a} = \sin^{-1} 0.81522146 = 0.95311148$$

$$h_{1b} = \sin^{-1} N_{h1b} = \sin^{-1} 0.83752057 = 0.99272961$$

$$h_{2a} = \sin^{-1} N_{h2a} = \sin^{-1} 0.83752057 = 0.99272961$$

$$h_{2b} = \sin^{-1} N_{h2b} = \sin^{-1} 0.83715326 = 0.99205772$$

$$R_{1a} = \frac{58.276}{3600 \times 180 / p} \frac{\sqrt{1 - 0.81522146^2}}{0.81522146} - \frac{0.0824}{3600 \times 180 / p} \left(\frac{\sqrt{1 - 0.81522146^2}}{0.81522146} \right)^3$$
$$= 0.00020057\, radian = 41.37''$$

$$R_{1b} = \frac{58.276}{3600 \times 180 / p} \frac{\sqrt{1 - 0.83752057^2}}{0.83752057} - \frac{0.0824}{3600 \times 180 / p} \left(\frac{\sqrt{1 - 0.83752057^2}}{0.83752057} \right)^3$$
$$= 0.00018421\, radian = 38.00''$$

$$R_{2a} = \frac{58.276}{3600 \times 180 / p} \frac{\sqrt{1 - 0.83752057^2}}{0.83752057} - \frac{0.0824}{3600 \times 180 / p} \left(\frac{\sqrt{1 - 0.83752057^2}}{0.83752057} \right)^3$$
$$= 0.00018421\, radian = 38.00''$$

$$R_{2b} = \frac{58.276}{3600\times 180/p}\frac{\sqrt{1-0.83715326^2}}{0.83715326} - \frac{0.0824}{3600\times 180/p}\left(\frac{\sqrt{1-0.83715326^2}}{0.83715326}\right)^3$$
$$= 0.00018448\,radian = 38.05''$$

From equation (5.5.1-27),

$$A = \cos(-H_{1b}) - \cos(-H_{1a}) = \cos(-6.21538725) - \cos(-5.99662377) = 0.038481147$$
$$B = \sin(-H_{1b}) - \sin(-H_{1a}) = \sin(-6.21538725) - \sin(-5.99662377) = -0.21490953$$
$$C = R_{1b}\frac{L_{h1b}\cos f\,\tan h_{1b} + \sin f\,\cos h_{1b}}{\cos d_1} - R_{1a}\frac{L_{h1a}\cos f\,\tan h_{1a} + \sin f\,\cos h_{1a}}{\cos d_1}$$

$$= 0.00018421\times\frac{0.54264618\times\cos 0.91028772\times\tan 0.99272961 + \sin 0.91028772\times\cos 0.99272961}{\cos 0.33466204}$$

$$- 0.00020057\times\frac{0.51394425\times\cos 0.91028772\times\tan 0.95311148 + \sin 0.91028772\times\cos 0.95311148}{\cos 0.33466204}$$

$$= 0.00018368 - 0.00019137 = -0.00000769$$

$$D = \cos(-H_{2b}) - \cos(-H_{2a}) = \cos(-6.35977114) - \cos(-6.21538725)$$
$$= -0.00063385367$$
$$E = \sin(-H_{2b}) - \sin(-H_{2a}) = \sin(-6.35977114) - \sin(-6.21538725)$$
$$= -0.14425712$$

$$F = R_{2b}\frac{L_{h2b}\cos f\tan h_{2b} + \sin f\cos h_{2b}}{\cos d_2} - R_{2a}\frac{L_{h2a}\cos f\tan h_{2a} + \sin f\cos h_{2a}}{\cos d_2}$$

$$= 0.00018448\times\frac{0.54217341\times\cos 0.91028772\times\tan 0.99205772 + \sin 0.91028772\times\cos 0.99205772}{\cos 0.33466204}$$

$$- 0.00018421\times\frac{0.54264618\times\cos 0.91028772\times\tan 0.99272961 + \sin 0.91028772\times\cos 0.99272961}{\cos 0.33466204}$$

$$= 0.00018380 - 0.00018368 = 0.00000012$$

From equations (5.5.1-29) and (5.5.1-30),

$$0.038481147u - 0.21490953v - 0.00000769 + 0.00016736 = 0$$

$$-0.00063385367u - 0.14425712v + 0.00000012 + 0.00031940 = 0$$

Solving the equations,

u = 0.008015 radian = 0.459^o = 1653"

v = 0.002178 radian = 0.125^o = 449"

Comparing these values with the result in section 5.5.2.1 (refraction neglected), effect of atmospheric refraction is small in this example.

5.6 Dome Slit Synchronization

Computer control of telescope has become popular in amateur astronomy, and now the advancement expands further. Dome slit control is an example of such advancement. In this section, equations to perform dome slit control for a telescope on German equatorial mount are derived. The equations are actually used by John Oliver of University of Florida to develop his dome control software "DomeSync".

The center of the telescope tube on German equatorial mount is offset from the center of a dome as shown in figure 5.6-1. Because of this, the azimuth of the dome slit is not coincident with the azimuth of the object which the telescope is aimed. The objective is to develop equations of the azimuth of the dome slit. This problem is a very good example of matrix method application.

Figure 5.6-1 shows definition of coordinate systems used in this section. Point O is the center of the dome. Point P is the intersection of the polar axis and the declination axis of the German equatorial mount. Point Q is the intersection of the telescope tube centerline and the declination axis. Dimensions of the dome and the mount are also defined in the figure.

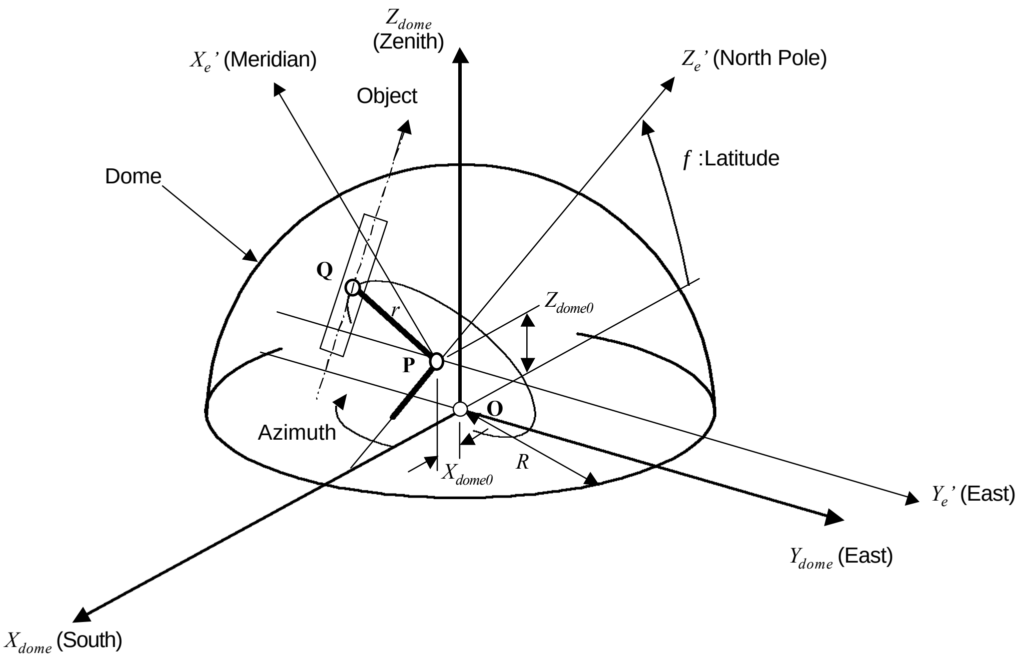
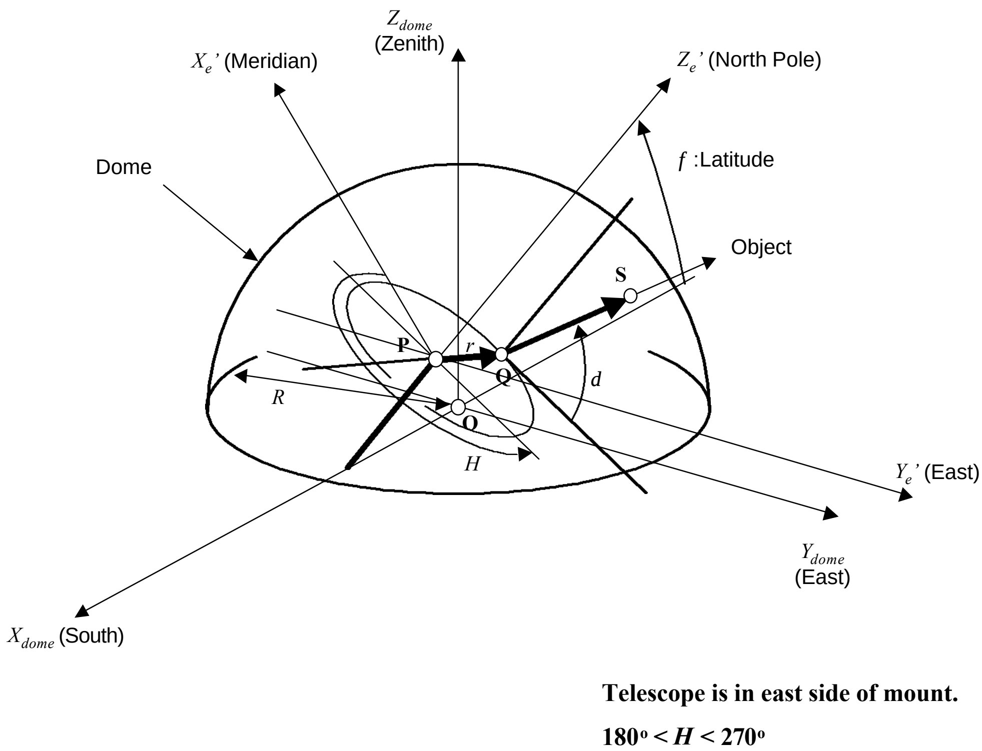


Figure 5.6-1 Definition of Coordinate Systems – Dome Slit Position

5.6.1 Object in First Quadrant

Equation when the object is in the first quadrant ($180^o < H < 270^o$) and telescope is in east side of mount is derived in this section. Point S in figure 5.6.1-1 is on the dome surface.



<u>Figure 5.6.1-1 Object in First Quadrant</u>

From equation (5.1-2), unit vector $\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|}$ is,

$$\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|} = \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix}$$

Vector P→Q in X_e'-Y_e'-Z_e' coordinate is,

$$\overrightarrow{PQ} = \begin{pmatrix} X_e' \\ Y_e' \\ Z_e' \end{pmatrix} = r \begin{pmatrix} \sin(H - p) \\ \cos(H - p) \\ 0 \end{pmatrix}$$

From equation (5.1-4), vector O→S in X_{dome}-Y_{dome}-Z_{dome} coordinate is,

$$\overrightarrow{OS} = \begin{pmatrix} X_{dome} \\ Y_{dome} \\ Z_{dome} \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & \sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \left(\begin{pmatrix} r\sin(H - p) \\ r\cos(H - p) \\ 0 \end{pmatrix} + k \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix} \right) + \begin{pmatrix} X_{dome0} \\ 0 \\ Z_{dome0} \end{pmatrix}$$

$$= \begin{pmatrix} X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\sin(H - p) + k\left(\cos\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d\right) \\ r\cos(H - p) + k\cos d \sin(-H) \\ Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\sin(H - p) + k\left(-\sin\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d\right) \end{pmatrix}$$

$$= \begin{pmatrix} A + Dk \\ B + Ek \\ C + Fk \end{pmatrix}$$

… Equation (5.6.1-1)

Where,

k: Constant

$$A = X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\sin(H - p)$$

$$B = r\cos(H - p)$$

$$C = Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\sin(H - p)$$

$$D = \cos\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d$$

$$E = \cos d \sin(-H)$$

$$F = -\sin\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d$$

… Equation (5.6.1-2)

5.6.2 Object in Second Quadrant

Object is in the second ($270^o < H < 360^o$) and telescope is in west side of mount.

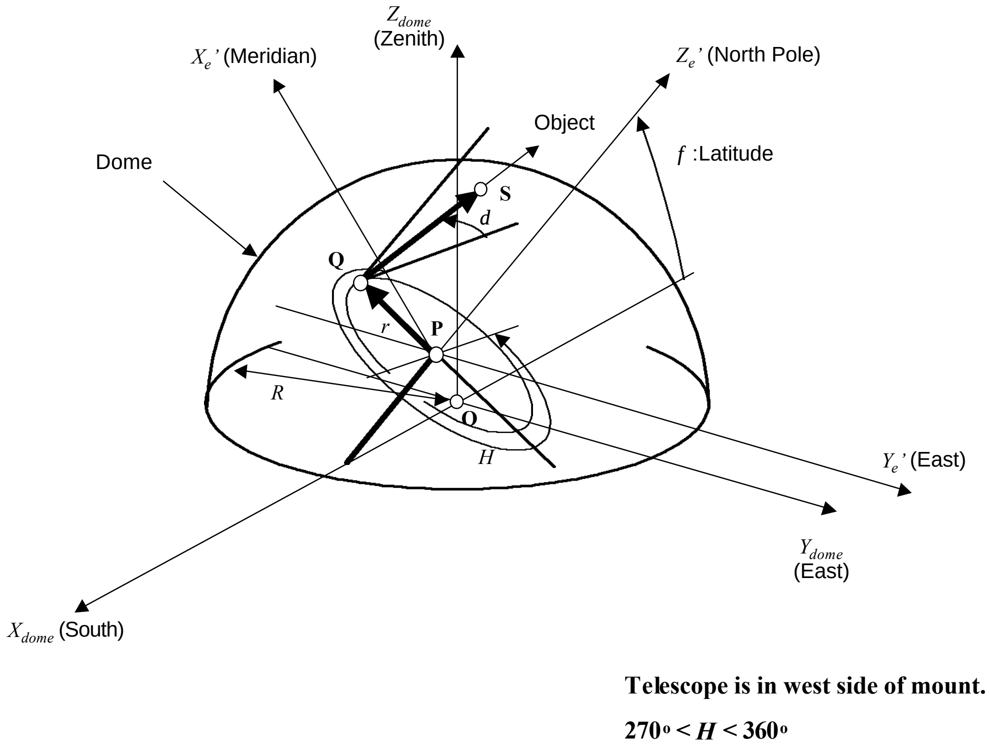


Figure 5.6.2-1 Object in Second Quadrant

From equation (5.1-2), unit vector $\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|}$ is,

$$\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|} = \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix}$$

Vector P→Q in X_e'-Y_e'-Z_e' coordinate is,

$$\overrightarrow{PQ} = \begin{pmatrix} X_e' \\ Y_e' \\ Z_e' \end{pmatrix} = r \begin{pmatrix} \cos\left(H - \frac{3p}{2}\right) \\ -\sin\left(H - \frac{3p}{2}\right) \\ 0 \end{pmatrix}$$

From equation (5.1-4), vector O→S in X_{dome}-Y_{dome}-Z_{dome} coordinate is,

$$\overrightarrow{OS} = \begin{pmatrix} X_{dome} \\ Y_{dome} \\ Z_{dome} \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & \sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \left(\begin{pmatrix} r\cos\left(H - \frac{3p}{2}\right) \\ -r\sin\left(H - \frac{3p}{2}\right) \\ 0 \end{pmatrix} + k \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix} \right) + \begin{pmatrix} X_{dome0} \\ 0 \\ Z_{dome0} \end{pmatrix}$$

$$= \begin{pmatrix} X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\cos\left(H - \frac{3p}{2}\right) + k\left(\cos\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d\right) \\ -r\sin\left(H - \frac{3p}{2}\right) + k\cos d \sin(-H) \\ Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\cos\left(H - \frac{3p}{2}\right) + k\left(-\sin\left(f - \frac{p}{2}\right)\cos d \cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d\right) \end{pmatrix}$$

$$= \begin{pmatrix} A + Dk \\ B + Ek \\ C + Fk \end{pmatrix}$$

… Equation (5.6.2-1)

Where,

k: Constant

$$A = X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\cos\left(H - \frac{3p}{2}\right)$$
$$B = -r\sin\left(H - \frac{3p}{2}\right)$$
$$C = Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\cos\left(H - \frac{3p}{2}\right)$$
$$D = \cos\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d$$
$$E = \cos d\,\sin(-H)$$
$$F = -\sin\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d$$

… Equation (5.6.2-2)

5.6.3 Object in Third Quadrant

Object is in the third quadrant ($0^o < H < 90^o$) and telescope is in east side of mount.

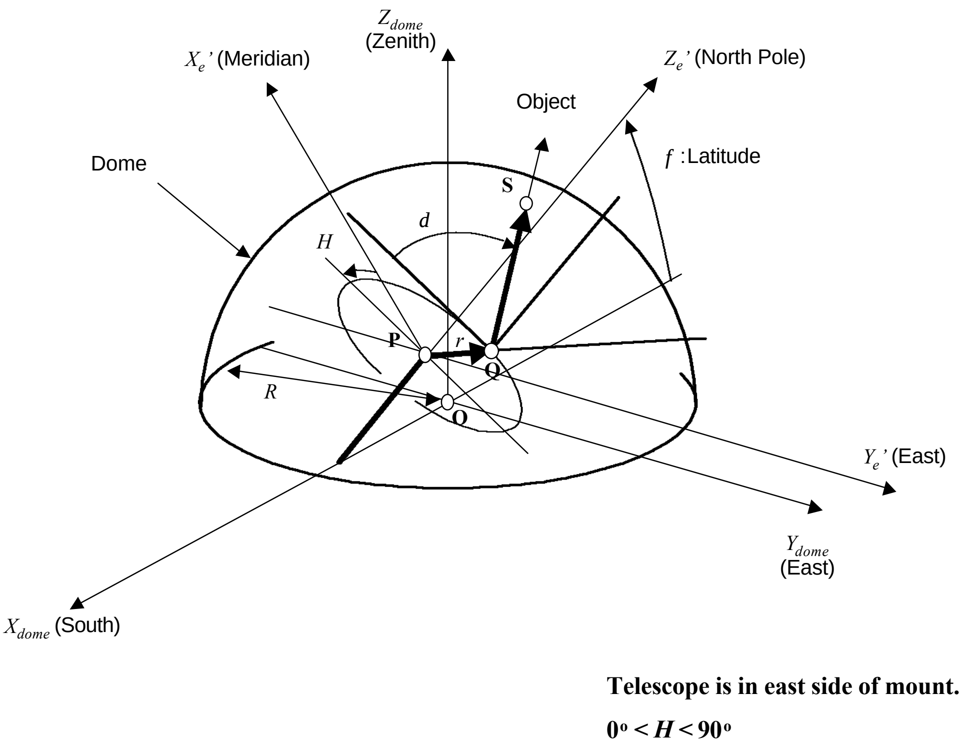


Figure 5.6.3-1 Object in Third Quadrant

From equation (5.1-2), unit vector $\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|}$ is,

$$\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|} = \begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix}$$

Vector P→Q in X_e'-Y_e'-Z_e' coordinate is,

$$\overrightarrow{PQ} = \begin{pmatrix} X_e' \\ Y_e' \\ Z_e' \end{pmatrix} = r \begin{pmatrix} \sin H \\ \cos H \\ 0 \end{pmatrix}$$

From equation (5.1-4), vector O→S in X_{dome}-Y_{dome}-Z_{dome} coordinate is,

$$\overrightarrow{OS} = \begin{pmatrix} X_{dome} \\ Y_{dome} \\ Z_{dome} \end{pmatrix} = \begin{bmatrix} \cos\left(f - \frac{p}{2}\right) & 0 & \sin\left(f - \frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f - \frac{p}{2}\right) & 0 & \cos\left(f - \frac{p}{2}\right) \end{bmatrix} \left(\begin{pmatrix} r \sin H \\ r \cos H \\ 0 \end{pmatrix} + k \begin{pmatrix} \cos d \cos(-H) \\ \cos d \sin(-H) \\ \sin d \end{pmatrix} \right) + \begin{pmatrix} X_{dome0} \\ 0 \\ Z_{dome0} \end{pmatrix}$$

$$= \begin{pmatrix} X_{dome0} + r \cos\left(f - \frac{p}{2}\right) \sin H + k\left(\cos\left(f - \frac{p}{2}\right) \cos d \cos(-H) + \sin\left(f - \frac{p}{2}\right) \sin d \right) \\ r \cos H + k \cos d \sin(-H) \\ Z_{dome0} - r \sin\left(f - \frac{p}{2}\right) \sin H + k\left(-\sin\left(f - \frac{p}{2}\right) \cos d \cos(-H) + \cos\left(f - \frac{p}{2}\right) \sin d \right) \end{pmatrix}$$

$$= \begin{pmatrix} A + Dk \\ B + Ek \\ C + Fk \end{pmatrix}$$

… Equation (5.6.3-1)

Where,

k: Constant

$$A = X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\sin H$$
$$B = r\cos H$$
$$C = Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\sin H$$
$$D = \cos\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d$$
$$E = \cos d\,\sin(-H)$$
$$F = -\sin\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d$$

… Equation (5.6.3-2)

5.6.4 Object in Fourth Quadrant

Object is in the fourth quadrant (90° < H < 180°) and telescope is in west side of mount.

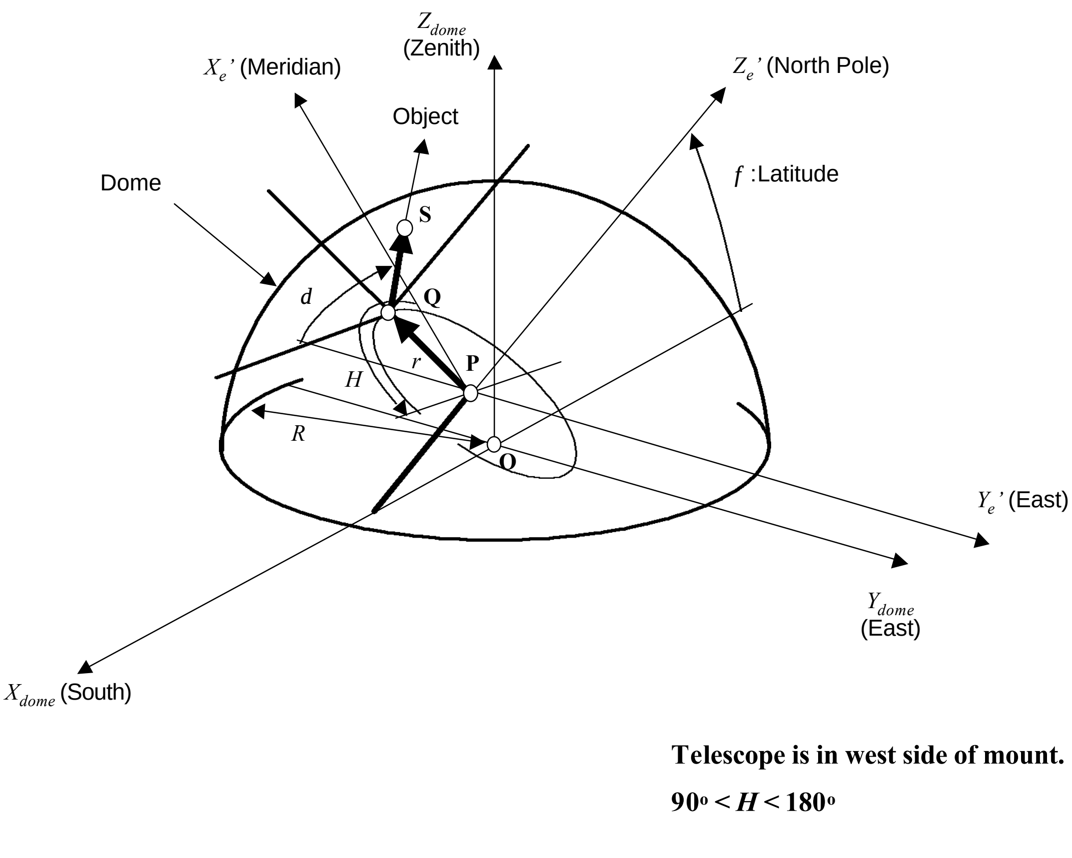


Figure 5.6.4-1 Object in Fourth Quadrant

From equation (5.1-2), unit vector $\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|}$ is,

$$\frac{\overrightarrow{QS}}{\left|\overrightarrow{QS}\right|}=\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix}=\begin{pmatrix} \cos d\cos(-H) \\ \cos d\sin(-H) \\ \sin d \end{pmatrix}$$

Vector P→Q in X_e'-Y_e'-Z_e' coordinate is,

$$\overrightarrow{PQ}=\begin{pmatrix} X_e' \\ Y_e' \\ Z_e' \end{pmatrix}=r\begin{pmatrix} \cos\left(H-\frac{p}{2}\right) \\ -\sin\left(H-\frac{p}{2}\right) \\ 0 \end{pmatrix}$$

From equation (5.1-4), vector O→S in X_{dome}-Y_{dome}-Z_{dome} coordinate is,

$$\overrightarrow{OS}=\begin{pmatrix} X_{dome} \\ Y_{dome} \\ Z_{dome} \end{pmatrix}=\begin{bmatrix} \cos\left(f-\frac{p}{2}\right) & 0 & \sin\left(f-\frac{p}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(f-\frac{p}{2}\right) & 0 & \cos\left(f-\frac{p}{2}\right) \end{bmatrix}\left(\begin{pmatrix} r\cos\left(H-\frac{p}{2}\right) \\ -r\sin\left(H-\frac{p}{2}\right) \\ 0 \end{pmatrix}+k\begin{pmatrix} \cos d\cos(-H) \\ \cos d\sin(-H) \\ \sin d \end{pmatrix}\right)+\begin{pmatrix} X_{dome0} \\ 0 \\ Z_{dome0} \end{pmatrix}$$

$$=\begin{pmatrix} X_{dome0}+r\cos\left(f-\frac{p}{2}\right)\cos\left(H-\frac{p}{2}\right)+k\left(\cos\left(f-\frac{p}{2}\right)\cos d\cos(-H)+\sin\left(f-\frac{p}{2}\right)\sin d\right) \\ -r\sin\left(H-\frac{p}{2}\right)+k\cos d\sin(-H) \\ Z_{dome0}-r\sin\left(f-\frac{p}{2}\right)\cos\left(H-\frac{p}{2}\right)+k\left(-\sin\left(f-\frac{p}{2}\right)\cos d\cos(-H)+\cos\left(f-\frac{p}{2}\right)\sin d\right) \end{pmatrix}$$

$$=\begin{pmatrix} A+Dk \\ B+Ek \\ C+Fk \end{pmatrix}$$

… Equation (5.6.4-1)

Where,

k: Constant

$$A = X_{dome0} + r\cos\left(f - \frac{p}{2}\right)\cos\left(H - \frac{p}{2}\right)$$

$$B = -r\sin\left(H - \frac{p}{2}\right)$$

$$C = Z_{dome0} - r\sin\left(f - \frac{p}{2}\right)\cos\left(H - \frac{p}{2}\right)$$

$$D = \cos\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \sin\left(f - \frac{p}{2}\right)\sin d$$

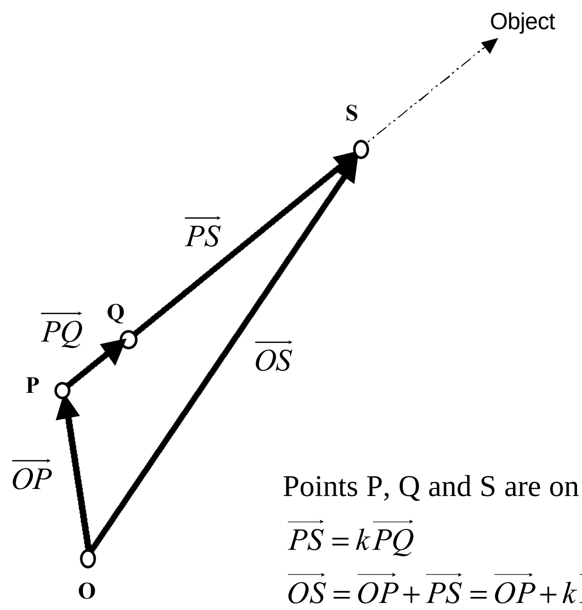
$$E = \cos d\,\sin(-H)$$

$$F = -\sin\left(f - \frac{p}{2}\right)\cos d\,\cos(-H) + \cos\left(f - \frac{p}{2}\right)\sin d$$

… Equation (5.6.4-2)

5.6.5 Intersection

Figure 5.6.5-1 shows vector expression of points on a straight line.



Points P, Q and S are on a straight line.

$$\overrightarrow{PS} = k\overrightarrow{PQ}$$

$$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \overrightarrow{OP} + k\overrightarrow{PQ}$$

Where k is a scalar.

Figure 5.6.5-1 Vector Expression of Points on a Straight Line

Since point S is on the dome surface, length of vector O→S is R.

$$(A + Dk)^2 + (B + Ek)^2 + (C + Fk)^2 = R^2$$

Then,

$$\left(D^2 + E^2 + F^2\right)k^2 + 2(AD + BE + CF)k - \left(R^2 - A^2 - B^2 - C^2\right) = 0$$

Solving this equation for k,

$$k = \frac{-(AD + BE + CF) + \sqrt{(AD + BE + CF)^2 + \left(D^2 + E^2 + F^2\right)\left(R^2 - A^2 - B^2 - C^2\right)}}{D^2 + E^2 + F^2}$$

… Equation (5.6.5-1)

$$\overrightarrow{OS} = \begin{pmatrix} X_{dome} \\ Y_{dome} \\ Z_{dome} \end{pmatrix} = \begin{pmatrix} A + Dk \\ B + Ek \\ C + Fk \end{pmatrix}$$

… Equation (5.6.5-2)

From equations (5.1-5) and (5.1-6), azimuth and elevation of dome slit are,

$$\tan(-A_{dome}) = \frac{Y_{dome}}{X_{dome}}$$

… Equation (5.6.5-3)

When $X_{dome} >= 0$, ($-A_{dome}$) is in the first quadrant or the fourth quadrant.

When $X_{dome} < 0$, ($-A_{dome}$) is in the second quadrant or the third quadrant.

Azimuth, A_{dome} is measured from south to westward.

$$\sin h_{dome} = \frac{Z_{dome}}{R}$$

… Equation (5.6.5-4)

$-p/2$ (-90^o) <= h_{dome} <= $+p/2$ ($+90^o$)